METRIC REALIZATION OF FUZZY SIMPLICIAL SETS

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Abstract. We discuss fuzzy simplicial sets, and their relationship to (a mild generalization of) metric spaces. Namely, we present an adjunction between the categories: a metric realization functor and fuzzy singular complex functor that generalize the usual geometric realization and singular functors. Finally, we show how these constructions relate to persistent homology.

The following document is a rough draft and may have (substantial) errors.

1. Fuzzy simplicial sets

Let I denote the topological space whose underlying set of points is $(0,1] \in \mathbb{R}$ and whose open sets are the intervals $(0,a)$ where $a \in (0,1]$. In other words, the category of open sets and inclusions in I is equivalent to the partially ordered set $(0,1]$ under the relation $\leq$. It has a Grothendieck topology given in the usual way, so I has an "underlying Grothendieck site." We may abuse notation by writing I to denote the poset of open sets in the topological space I, either as a category or as a site.

A sheaf $S \in \mathbf{Shv}(I)$ on I is a functor $S\colon I^{\mathrm{op}} \to \mathbf{Sets}$ satisfying the sheaf condition. Explicitly, S consists of a set $S([0,a))$ for all $a \in (0,1]$, which we choose to denote by $S^{\geq a}$, and restriction maps $\rho_{b,a}\colon S^{\geq b} \to S^{\geq a}$ for all $b \geq a$, such that if $c \geq b \geq a$ then $\rho_{b,a} \circ \rho_{c,b} = \rho_{c,a}$, and such that, for any non-empty subset $A \subset (0,1]$ with supremum $a = \sup(A)$, one has

$$S^{\geq a} \cong \lim_{a' \in A} S^{\geq a'}.$$

A sheaf S is called a *fuzzy set* if for each $b \geq a$ in $(0,1]$, the restriction map $\rho_{b,a}$ is injective. Let $\mathbf{Fuz}$ denote the full subcategory of $\mathbf{Shv}(I)$ spanned by the fuzzy sets. This definition is slightly different than Goguen's [Gog], but is closely related. See [Bar]. The difference between fuzzy sets T and arbitrary sheaves $S \in \mathbf{Shv}(I)$ is that, in T two elements are either equal or they are not, whereas two elements $x \neq y \in S^{\leq a}$ may be equal *to a certain degree*, $\rho_{a,c}(x) = \rho_{a,c}(y)$ for some $c < a$.

Suppose $S \in \mathbf{Shv}(I)$ is a sheaf. For $a \in (0,1]$, let $S(a) = S^{\geq a} - \mathrm{colim}_{b>a} S^{\geq b}$, and note that $S^{\geq a} = \mathrm{colim}_{b \geq a} \rho_{b,a}[S(b)]$. If T is a fuzzy set, we can make this easier on the eyes:

$$T^{\geq a} = \coprod_{b \geq a} T(b).$$

We write $x \in S$ and say that *x is an element of S*, if there exists $a \in (0,1]$ such that $x \in S(a)$; in this case we may say that *x is an element of S with strength a.*

The following definition may bring things down to earth.

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Definition 1.1. A *classical fuzzy set* is a pair (X,η) where X is a set and $\eta\colon X\to(0,1]$ is a function. Given classical fuzzy sets (X,η_X) and (Y,η_Y), *morphism of classical fuzzy sets* between them is a function $f\colon X\to Y$ such that $\eta_Y(f(x))\geq\eta_X(x)$ for all $x\in X$.

Let T denote a fuzzy set. Let $\bar{T}$ denote the set $\bigcup_{a\in(0,1]}T^{\geq a}$, and define $\eta\colon\bar{T}\to(0,1]$ by $\eta(x)=a$ if the following condition holds: $x\in T^{\geq a}$ and for all $a'\geq a$ one has $x\notin T^{\geq a'}$. We refer to the pair $(\bar{T},\eta)$ as the *classical form for* T. We refer to $\bar{T}$ as the *underlying set* of T and to η as the *characteristic function* for T. This construction is functorial and induces an isomorphism of categories between fuzzy sets and classical fuzzy sets.

The following lemma says that, under a map of fuzzy sets, an element cannot be sent to an element of lower strength.

Lemma 1.2. *Suppose that S and T are fuzzy sets. If $f\colon S\to T$ is a morphism of fuzzy sets, then for all $a,b\in(0,1]$, if $x\in S(a)$ then $f(x)\in T(b)$ for some $b\geq a$.*

Proof. Since $x\in S^{\geq a}$, we have by definition that $f(x)\in T^{\geq a}$, so $x\in T(b)$ for some $b\geq a$.

□

Lemma 1.3. *The forgetful functor* **Fuz** $\to$ **Shv**(I) *is fully faithful and has a left adjoint m. Thus* **Fuz** *is closed under taking colimits.*

Proof. Given a sheaf $S\colon I^{\mathrm{op}}\to\mathbf{Sets}$ and $a\in(0,1]$, let $(mS)^{\geq a}=S^{\geq a}/\sim$, where for $x,x'\in S^{\geq a}$, we set $x\sim x'$ if there exists $b\leq a$ such that $\rho_{a,b}(x)=\rho_{a,b}(x')$. Clearly, mS is a fuzzy set, and one checks that m is left adjoint to the forgetful functor.

To compute the colimit of a diagram in **Fuz**, one applies the forgetful functor, takes the colimit in **Shv**(I), and applies the left adjoint.

□

Let $\mathbf{\Delta}$ denote the simplicial indexing category—i.e. the category of nonempty finite ordered sets—and denote its objects by $[n]$ for $n\in\mathbb{N}$.

Definition 1.4. A *fuzzy simplicial set* is a functor $\mathbf{\Delta}^{\mathrm{op}}\to\mathbf{Fuz}$. A *morphism of fuzzy simplicial sets* is a natural transformation of functors. The category of fuzzy simplicial sets is denoted **sFuz**.

A fuzzy simplicial set is a simplicial set in which every simplex has a strength. A simplex has strength at most the minimum of its faces. All degeneracies of a simplex have the same strength as the simplex.

A fuzzy simplicial set $X\colon\mathbf{\Delta}^{\mathrm{op}}\to\mathbf{Fuz}$ can be rewritten as a sheaf $X\colon(\mathbf{\Delta}\times I)^{\mathrm{op}}\to\mathbf{Sets}$, where $\mathbf{\Delta}$ has the trivial Grothendieck topology and $\mathbf{\Delta}\times I$ has the product Grothendieck topology. We write $X_n^{\geq a}$ to denote the set $X([n],[0,a))$.

For $n\in\mathbb{N}$ and $i\in I$, let $\Delta_i^n\in\mathbf{sFuz}$ denote the functor represented by (n,i). If $i=[0,a)$ we may also write $\Delta_{<a}^n$ to denote Δ_i^n. Note that a map $f\colon[n]\to[m]$ induces a unique map $F\colon\Delta_{<a}^n\to\Delta_{<b}^m$ if and only if $a\leq b$; otherwise there can be no such F.

Any fuzzy simplicial set X can be canonically written as the colimit of its diagram of simplices:

$$\operatorname*{colim}_{\Delta_{<a}^n\to X}\Delta_{<a}^n\xrightarrow{\cong}X$$

2. UBER-METRIC SPACES

We define a category of uber-metric spaces, which are metric spaces except with the possibility of $d(x,y)=\infty$ or $d(x,y)=0$ for $x \neq y$.

Definition 2.1. An *uber-metric space* is a pair (X,d), where X is a set and $d\colon X \times X \to [0,\infty]$, such that for all $x,y,z \in X$,

(1) $d(x,x)=0$,
(2) $d(x,y)=d(y,x)$, and
(3) $d(x,z) \leq d(x,y)+d(y,z)$.

Here we consider $x \leq \infty$ and $x+\infty=\infty+x=\infty$ for all $x \in [0,\infty]$. We call d an *uber-metric* or just a *metric* on X.

A *morphism of uber-metric spaces*, denoted $f\colon (X,d_X) \to (Y,d_Y)$ is a function $f\colon X \to Y$ such that $d_Y(f(x_1),f(x_2)) \leq d_X(x_1,x_2)$ for all $x_1,x_2 \in X$. Such functions are also called *non-expansive*.

These objects and morphisms define a category called *the category of uber-metric spaces* and denoted **UM**.

Lemma 2.2. *The category* **UM** *is closed under colimits.*

Proof. We must show that **UM** has an initial object, arbitrary coproducts, and coequalizers. The set $\emptyset$ is the initial object in **UM**.

Let A be a set and for all $a \in A$, let (X_a,d_a) denote a metric space. Let X_A denote the set $\coprod_{a\in A} X_a$; and let d_A denote the metric such that for all $y,y' \in X_A$, if there exists $a \in A$ such that $y,y' \in X_a$ then $d_A(y,y')=d_a(y,y')$, but if instead y and y' are in separate components then $d_A(y,y')=\infty$. One checks that (X_A,d_A) is an uber-metric space and that it satisfies the universal property for a coproduct.

Finally, suppose that

$$A \underset{g}{\overset{f}{\rightrightarrows}} X \xrightarrow{[-]} Y$$

is a coequalizer diagram of sets. Write $x \sim x'$ if there exists $a \in A$ with $x = f(a), y=g(a)$; then $Y = X/\sim$ is the set of equivalence classes. If d_X is a metric on X, we define a metric ([Wik]) d_Y on Y by

$$d_Y([x],[x'])=\inf_{n,p,q}(d_X(p_1,q_1)+d_X(p_2,q_2)+\cdots+d_X(p_n,q_n)),$$

where the infimum is taken over all pairs of sequences $(p_1,\ldots,p_n),(q_1,\ldots,q_n)$ of elements of X, such that $p_1 \sim x$, $q_n \sim x'$, and $p_{i+1} \sim q_i$ for all $1 \leq i \leq n-1$. Again, one checks that (Y,d_Y) is an uber-metric space which satisfies the universal property of a coequalizer.

□

3. METRIC REALIZATION

In order to define a metric realization functor $Re\colon \mathbf{sFuz} \to \mathbf{UM}$, we first define it on the representable sheaves in **sFuz** and then extend to the whole category using colimits (i.e. using a left Kan extention).

Recall the usual metric on Euclidean space $\mathbb{R}^m$ and let $\mathbb{R}^m_{\geq 0}$ denote the m-tuples all of whose entries are non-negative. Recall also that objects of I are of the form $[0,a)$ for $0<a\leq 1$. For an object $([n],[0,a)) \in \mathbb{N}\times I$, define $Re(\Delta^n_{<a})$, as a set, to be

$$\{(x_0,x_1,\ldots,x_n) \subset \mathbb{R}^{n+1} | x_0+x_1+\cdots+x_n = -\lg(a)\},$$

where $\lg(a)$ means the log (base 2) of a. We take as our metric on $Re(\Delta^n_{<a})$ to be that induced by its inclusion as a subspace of $\mathbb{R}^{n+1}$.

A morphism $([n], [0, a)) \to ([m], [0, b))$ exists if $a \le b$, and in that case consists of a morphism $\sigma\colon [n] \to [m]$. We define $Re(\sigma, a \le b)\colon Re(\Delta^n_{<a}) \to Re(\Delta^m_{<b})$ to be the map

$$(x_0, x_1, \ldots, x_n) \mapsto \frac{\lg(b)}{\lg(a)} \left(\sum_{i_0 \in \sigma^{-1}(0)} x_{i_0}, \sum_{i_1 \in \sigma^{-1}(1)} x_{i_1}, \ldots, \sum_{i_m \in \sigma^{-1}(m)} x_{i_m} \right).$$

Note that this map is non-expansive because $0 < a \le b \le 1$ implies that $\frac{\lg(b)}{\lg(a)} \le 1$.

We are ready to define Re on a general X as

$$Re(X) := \operatorname*{colim}_{\Delta^n_{<a} \to X} Re(\Delta^n_{<a}).$$

This functor preserves colimits, so it has a right adjoint, which we denote $Sing\colon \mathbf{UM} \to \mathbf{sFuz}$. It is given on $Y \in \mathbf{UM}$ by

$$Sing(Y)^{<a}_n = \mathrm{Hom}_{\mathbf{UM}}(Re(\Delta^n_{<a}), Y).$$

4. Persistent Homology

Let $Ch_{\geq 0}$ denote the category of non-negatively graded chain complexes of abelian groups (whose differential decreases degree). Given a fuzzy simplicial set X, its *persistence complex* is a functor $P_X\colon I^{\mathrm{op}} \to Ch_{\geq 0}$, which in degree n is

$$P_X([0, a))_n = \mathbb{Z}\langle X^{\geq a}_n \rangle,$$

the free $\mathbb{Z}$-module generated by the n-simplices of strength at least a in X. The boundary maps are computed in the usual way. The homology of P_X is the functor

$$H_*(-; \mathbb{Z}) \circ P_X\colon I^{\mathrm{op}} \to \mathbf{gVect},$$

where $\mathbf{gVect}$ is the category of graded vector spaces. In other words, for every $a \in (0, 1]$, one has homology groups $H^{\geq a}_n(X; \mathbb{Z})$, and if $a \geq b$, one has a map $H^{\geq a}_n(X; \mathbb{Z}) \to H^{\geq b}_n(X; \mathbb{Z})$.

Given a finite metric space (M, d_M), one can form a fuzzy graph $G(M)$ as follows. On vertices, take $G(M)^{\geq a}_0 = M$, for all $a \in (0, 1]$. For edges, take

$$G(M)^{\geq a}_1 = \{(m_1, m_2) \in M \times M | d_M(m_1, m_2) \le -\lg(a)\}$$

Now of course, there is a left adjoint that builds a fuzzy simplicial set $F(G)$ out of a fuzzy graph G, using cliques. Precisely, we have

$$F(G)^{\geq a}_n = \mathrm{Hom}(sk_1(\Delta^n_{\le a}), G),$$

where sk_1 denotes the 1-skeleton functor from fuzzy simplicial sets to fuzzy graphs.

Taking our original finite metric space M, form $F(G(M))$, take its persistent homology in the above sense.

Question 4.1. and I wonder if this agrees with its persistent homology in the sense I heard about from Gunnar Carlsson.

Question 4.2. Is there any reasonable way in which we could take a homotopy colimit of our fuzzy simplicial set $X\colon (\mathbf{\Delta} \times I)^{\mathrm{op}} \to \mathbf{Sets}$ and get back something reasonable? (For simplicial sets Y, the homotopy colimit of the composite $Y\colon \mathbf{\Delta} \to \mathbf{Sets} \to \mathbf{Top}$ gives the "right" answer, the geometric realization of Y.)

Appendix A. Probabilistic simplicial sets

One might want to consider fuzzy simplicial sets X in the following way: for each n-simplex $x \in X_n$, consider $\eta_n(x) \in (0,1]$ as "the probability that x is in X." Now simulate X by writing down lots of (non-fuzzy) simplicial sets X' whose simplices are chosen from X with the indicated probabilities.

Unfortunately, this does not make much sense. For example, consider the fuzzy simplex $\Delta^1_{\leq .5}$. This has two 0-simplices and a single 1-simplex, each with value $\frac{1}{2}$. To simulate it, one would flip a coin to decide whether to include each vertex and then, only if both vertices were included, flip another coin to decide whether to include the 1-simplex. Thus the 1-simplex is in the simulated simplicial set with probability $\frac{1}{8}$ – this does not seem like a good notion, because the natural understanding of $\Delta^1_{\leq .5}$ is that each of the three simplices has a probability of $\frac{1}{2}$, whatever that might mean.

Notice that $M := (0,1]$ is a monoid under multiplication.

Definition A.1. Let $\mathcal{X} = (X, \eta_X)$ and $\mathcal{Y}(Y, \eta_Y)$ denote fuzzy sets. We define their *tensor product* $\mathcal{X} \otimes \mathcal{Y}$ to be the fuzzy set $(X \times Y, \eta)$, where $\eta(x,y) = \eta_X(x)\eta_Y(y)$ for each $x \in X, y \in Y$. There is an obvious map of fuzzy sets $\mathcal{X} \otimes \mathcal{Y} \to \mathcal{X} \times \mathcal{Y}$.

Let X denote a fuzzy simplicial set. Recall that the face maps induce, for every $n \in \mathbb{N}$, a map of fuzzy sets $d\colon X_n \to X_{n-1} \times \cdots \times X_{n-1}$, where there are $n+1$ copies in the product.

Definition A.2. A *probabilistic simplicial set* is a fuzzy simplicial set X such that for each integer $n \geq 1$ there exists a dotted lift in the diagram

$$\begin{array}{ccc} & & (X_{n-1})^{\otimes n+1} \\ & \nearrow & \downarrow \\ X_n & \xrightarrow[d]{} & (X_{n-1})^{\times n+1}. \end{array}$$

In other words, the value of an n-simplex in a probabilistic simplicial set is at most the product of the values of its faces. Now we can define an appropriate concept of simulation.

(Caveat: my understanding of probability theory is limited, and this will be obvious when reading the following definition.)

Definition A.3. Given a set X and a function $\eta\colon X \to (0,1]$, an *simulation* of (X, η) is a subset $X' \subset X$ such that for each $x \in X$, the probability that $x \in X'$ is $\eta(x)$.

Definition A.4. Let (X, η) denote a fuzzy simplicial set, and for each $n \in \mathbb{N}$, let $\eta_n\colon X_n \to (0,1]$ denote the characteristic function of level n. A *simulation* of (X, η) is a simplicial set X' with the following properties:

- The set X'_0 of 0-simplices is a simulation of (X_0, η_0)
- Let X''_1 denote the fiber product in the diagram of sets

$$\begin{array}{ccc} X''_1 & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X'_0 \times X'_0 & \longrightarrow & X_0 \times X_0, \end{array}$$

the set of *possible 1-simplices given the simulation* X_0', and let $\eta_1''\colon X_1'' \to (0,1]$ be defined by $\eta_1''(x) = \frac{\eta_1(x)}{\eta_0(d_0(x))\cdot\eta_0(d_1(x))}$ for each $x \in X_1''$. Now define X_1' to be a simulation of X_1''.

- Inductively, we can define a fuzzy set (X_n'', η_n'') given a simulation X_{n-1}' of the previous stage, and then simulate it as X_n'.

Question A.5. Suppose that X is a probabilistic simplicial set. There are two ways to imagine what should be the Betti numbers of X. The first is to take Betti numbers of the persistence complex $P_X([0,a))$ as a function of $a \in (0,1]$, and then compute the expected values – i.e. integrate this function over the unit interval. The second way is to simulate the fuzzy simplicial set X and take averages. Is there any relationship between these approaches?

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