METRIC REALIZATION OF FUZZY SIMPLICIAL SETS

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Abstract. We discuss fuzzy simplicial sets, and their relationship to something like metric spaces. Namely, we present an adjunction between the categories: a metric realization functor and fuzzy singular complex functor.

The following document is a rough draft and may have (substantial) errors.

1. Fuzzy simplicial sets

Let I denote the Grothendieck site whose objects are initial open intervals contained in the half-open unit interval $[0,1) \in \mathbb{R}$, whose morphisms are inclusions of open subsets, and whose covers are open covers. In other words, as a category, I is equivalent to the partially ordered set $(0,1]$ under the relation $\leq$.

A sheaf $S \in \mathbf{Shv}(I)$ on I is a functor $S\colon I^{\mathrm{op}} \to \mathbf{Sets}$ satisfying the sheaf condition. Explicitly, S consists of a set $S([0,a))$ for all $a \in (0,1]$, which we choose to denote by $S^{\geq a}$, and restriction maps $\rho_{b,a}\colon S^{\geq b} \to S^{\geq a}$ for all $b \geq a$, such that if $c \geq b \geq a$ then $\rho_{b,a} \circ \rho_{c,b} = \rho_{c,a}$, and such that for all $a \in I$, one has

$$S^{\geq a} \cong \lim_{a'<a} S^{\geq a'}.$$

A sheaf S is called a *fuzzy set* if for each $b \geq a$ in $(0,1]$, the restriction map $\rho_{b,a}$ is injective. Let $\mathbf{Fuz}$ denote the full subcategory of $\mathbf{Shv}(I)$ spanned by the fuzzy sets. This definition is slightly different than Goguen's [**?**], but is closely related. See [**?**]. The difference between fuzzy sets T and arbitrary sheaves $S \in \mathbf{Shv}(I)$ is that, in T two elements are either equal or they are not, whereas two elements $x \neq y \in S^{\leq a}$ may be equal *to a certain degree*, $\rho_{a,c}(x) = \rho_{a,c}(y)$ for some $c < a$.

Suppose $S \in \mathbf{Shv}(I)$ is a sheaf. For $a \in (0,1]$, let $S(a) = S^{\geq a} - \mathrm{colim}_{b>a} S^{\geq b}$, and note that $S^{\geq a} = \mathrm{colim}_{b \geq a}\, \rho_{b,a}[S(b)]$. If T is a fuzzy set, we can make this easier on the eyes:

$$T^{\geq a} = \coprod_{b \geq a} T(b).$$

We write $x \in S$ and say that *x is an element of S*, if there exists $a \in (0,1]$ such that $x \in S(a)$; in this case we may say that *x is an element of S with strength a.*

The following lemma says that, under a map of fuzzy sets, an element cannot be sent to an element of lower strength.

Lemma 1.1. *Suppose that S and T are fuzzy sets. If $f\colon S \to T$ is a morphism of fuzzy sets, then for all $a, b \in (0,1]$, if $x \in S(a)$ then $f(x) \in T(b)$ for some $b \geq a$.*

Proof. Since $x \in S^{\geq a}$, we have by definition that $f(x) \in T^{\geq a}$, so $x \in T(b)$ for some $b \geq a$.

□

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Lemma 1.2. *The forgetful functor* **Fuz** $\to$ **Shv**(I) *is fully faithful and has a left adjoint* m*. Thus* **Fuz** *is closed under taking colimits.*

Proof. Given a sheaf $S\colon I^{\mathrm{op}} \to$ **Sets** and $a \in (0,1]$, let $(mS)^{\geq a} = S^{\geq a}/\sim$, where for $x, x' \in S^{\geq a}$, we set $x \sim x'$ if there exists $b \leq a$ such that $\rho_{a,b}(x) = \rho_{a,b}(x')$. Clearly, mS is a fuzzy set, and one checks that m is left adjoint to the forgetful functor.

To compute the colimit of a diagram in **Fuz**, one applies the forgetful functor, takes the colimit in **Shv**(I), and applies the left adjoint. □

Let $\mathbf{\Delta}$ denote the simplicial indexing category, and denote its objects by $[n]$ for $n \in \mathbb{N}$.

Definition 1.3. A *fuzzy simplicial set* is a functor $\mathbf{\Delta}^{\mathrm{op}} \to$ **Fuz**. A *morphism of fuzzy simplicial sets* is a natural transformation of functors. The category of fuzzy simplicial sets is denoted **sFuz**.

A fuzzy simplicial set is a simplicial set in which every simplex has a strength. A simplex has strength at most the minimum of its faces. All degeneracies of a simplex have the same strength as the simplex.

A fuzzy simplicial set $X\colon \mathbf{\Delta}^{\mathrm{op}} \to$ **Fuz** can be rewritten as a sheaf $X\colon (\mathbf{\Delta}\times I)^{\mathrm{op}} \to$ **Sets**, where $\mathbf{\Delta}$ has the trivial Grothendieck topology and $\mathbf{\Delta} \times I$ has the product Grothendieck topology. We write $X_n^{<a}$ to denote the set $X([n], [0,a))$.

For $n \in \mathbb{N}$ and $i \in I$, let $\Delta_i^n \in$ **sFuz** denote the functor represented by (n, i). If $i = [0, a)$ we may also write $\Delta_{<a}^n$ to denote Δ_i^n. Note that a map $f\colon [n] \to [m]$ induces a unique map $F\colon \Delta_{<a}^n \to \Delta_{<b}^m$ if and only if $a \leq b$; otherwise there can be no such F.

Any fuzzy simplicial set X can be canonically written as the colimit of its diagram of simplices:

$$\operatorname*{colim}_{\Delta_{<a}^n \to X} \Delta_{<a}^n \xrightarrow{\cong} X$$

2. UBER-METRIC SPACES

We define a category of uber-metric spaces, which are metric spaces except with the possibility of $d(x,y) = \infty$ or $d(x,y) = 0$ for $x \neq y$.

Definition 2.1. An *uber-metric space* is a pair (X, d), where X is a set and $d\colon X \times X \to [0, \infty]$, such that for all $x, y, z \in X$,

(1) $d(x,x) = 0$,
(2) $d(x,y) = d(y,x)$, and
(3) $d(x,z) \leq d(x,y) + d(y,z)$.

Here we consider $x \leq \infty$ and $x + \infty = \infty + x = \infty$ for all $x \in [0,\infty]$. We call d an *uber-metric* or just a *metric* on X.

A *morphism of uber-metric spaces*, denoted $f\colon (X, d_X) \to (Y, d_Y)$ is a function $f\colon X \to Y$ such that $d_Y(f(x_1), f(x_2)) \leq d_X(x_1, x_2)$ for all $x_1, x_2 \in X$. Such functions are also called *non-expansive*.

These objects and morphisms define a category called *the category of uber-metric spaces* and denoted **UM**.

Lemma 2.2. *The category* **UM** *is closed under colimits.*

Proof. We must show that **UM** has an initial object, arbitrary coproducts, and coequalizers. The set $\emptyset$ is the initial object in **UM**.

Let A be a set and for all $a \in A$, let (X_a, d_a) denote a metric space. Let X_A denote the set $\coprod_{a\in A} X_a$; and let d_A denote the metric such that for all $y, y' \in X_A$, if there exists $a \in A$ such that $y, y' \in X_a$ then $d_A(y, y') = d_a(y, y')$, but if instead y and y' are in separate components then $d_A(y, y') = \infty$. One checks that (X_A, d_A) is an uber-metric space and that it satisfies the universal property for a coproduct.

Finally, suppose that

$$A \underset{g}{\overset{f}{\rightrightarrows}} X \xrightarrow{[-]} Y$$

is a coequalizer diagram of sets. Write $x \sim x'$ if there exists $a \in A$ with $x = f(a), y = g(a)$; then $Y = X/\sim$ is the set of equivalence classes. If $y = m(x)$. If d_X is a metric on X, we define a metric ([**?**]) d_Y on Y by

$$d_Y([x], [x']) = \inf(d_X(p_1, q_1) + d_X(p_2, q_2) + \cdots + d_X(p_n, q_n)),$$

where the infemum is taken over all pairs of sequences $(p_1, \ldots, p_n), (q_1, \ldots, q_n)$ of elements of X, such that $p_1 \sim x$, $q_n \sim x'$, and $p_{i+1} \sim q_i$ for all $1 \leq i \leq n-1$. Again, one checks that (Y, d_Y) is an uber-metric space which satisfies the universal property of a coequalizer.

□

3. Metric realization

In order to define a metric realization functor $Re\colon \mathbf{sFuz} \to \mathbf{UM}$, we first define it on the representable sheaves in **sFuz** and then extend to the whole category using colimits (i.e. using a left Kan extention).

Recall the usual metric on Euclidean space $\mathbb{R}^m$ and let $\mathbb{R}^m_{\geq 0}$ denote the m-tuples all of whose entries are non-negative. Recall also that objects of I are of the form $[0, a)$ for $0 < a \leq 1$. For an object $([n], [0, a)) \in \mathbb{N} \times I$, define $Re(\Delta^n_{<a})$, as a set, to be

$$\{(x_0, x_1, \ldots, x_n) \subset \mathbb{R}^{n+1} | x_0 + x_1 + \cdots + x_n = 1 - a\}$$

We take as our metric on $Re(\Delta^n_{<a})$ to be that induced by its inclusion as a subspace of $\mathbb{R}^{n+1}$.

A morphism $([n], [0, a)) \to ([m], [0, b))$ exists if $a \leq b$, and in that case consists of a morphism $\sigma\colon [n] \to [m]$. We define $Re(\sigma, a \leq b)\colon Re(\Delta^n_{<a}) \to Re(\Delta^m_{<b})$ to be the map

$$(x_0, x_1, \ldots, x_n) \mapsto \frac{1-b}{1-a}\left(\sum_{i_0 \in \sigma^{-1}(0)} x_{i_0}, \sum_{i_1 \in \sigma^{-1}(1)} x_{i_1}, \ldots, \sum_{i_m \in \sigma^{-1}(m)} x_{i_m}\right).$$

Note that this map is non-expansive because $1 - b \leq 1 - a$.

We are ready to define Re on a general X as

$$Re(X) := \operatorname*{colim}_{\Delta^n_{<a} \to X} Re(\Delta^n_{<a}).$$

This functor preserves colimits, so it has a right adjoint, which we denote $Sing\colon \mathbf{UM} \to \mathbf{sFuz}$. It is given on $Y \in \mathbf{UM}$ by

$$Sing(Y)^{<a}_n = \mathrm{Hom}_{\mathbf{UM}}(Re(\Delta^n_{<a}), Y).$$

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