Project Narrative: Categorical approach to agent interaction

David I. Spivak

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Abstract

I am pursuing a category-theoretic model for the study of interaction. In particular I will consider the following sort of system: a network of interacting entities (or agents), such that each entity is an instantiation of a given type of structure and each interaction follows some mathematically prescribed rules. This research applies to a wide variety of systems found in the natural world and in technology. Indeed, the nervous system, human society, hierarchical protein materials, computers and networks, and complex organizations such as businesses and militaries are all systems that can be modeled in this way. It also has direct application to problems of information integration and distributed intelligence.

Statement of objectives

As stated in the introduction (Section 1.1), I am pursuing a mathematical model for the study of interaction. In particular I will consider the following sort of system: a network of individuals entities (or agents), each of which is an instantiation of a given type of structure, wherein these entities interact in a prescribed manner. Such systems are ubiquitous in the natural world and in technology. Indeed, the nervous system, human society, hierarchical protein materials, computers and networks, and complex organizations such as businesses and militaries are very different systems that can all be modeled in this way. Understanding these interactions and their exchanged semantics is important for fusing dynamic information from distributed agents.

This research will include two main aspects. The first is a detailed look at individual interactions, in which two agents are tasked with developing a richer medium for communicating with each other. To begin this requires carefully formulating the structure of the information which get passed between agents, the way such information is received, and the sense in which the agents are modified by successful receipt of new information. The second is an investigation of the relationship between small-scale interactions and their global aggregates. For example, given a complete understanding of the stimulus-response pattern for each neuron and of the connection pattern of neurons, I will provide a formula for predicting the stimulus-response pattern of the whole system.

I will discuss these two themes in order: In Section 2.1, I will propose a category-theoretic "communication protocol" by which agents can discuss and learn from one another, and in Section 2.2, I will outline an approach to modeling the relationship between individual interactions and their results in aggregate.

Chapter 1

Introduction

1.1 Project summary

The relationship between *structure* and *function* remains one of the most fascinating and important frontiers of research, spanning multiple disciplines. In neuroscience, the system of interconnected neurons in the brain is a structure whose function includes both rudimentary reaction to stimuli and intricate thought processes. In linguistics, the syntactical structures in words and sentences yield arbitrarily complex semantic meanings. In computer programs, the arrangement of simple modules (or even transistors) yields diverse applications. And in sociology, the network of human interaction and communication leads to large-scale social movements and political decisions.

Researchers in a variety of fields are finding that complex interactions between individuals within a single system can lead to exceptional qualities in aggregate. Such phenomena occur at multiple scales, and changes in the micro-scale not only affect but are affected by changes at the meso- or macro-scale. Our whole scientific paradigm is based on the idea that the unending complexity of the world around us is fundamentally correlated to the interactions of atomic building blocks.

The strength of spider silk and the capacities of the human brain lead devoted researchers to analyze these systems in an effort to locate where the magic is. That is, we are compelled to determine what is responsible for the abundance of diversity in the world, given the simplicity of its elementary units. It has been shown [KXI] that inexpensive and abundant atomic building blocks, when well-organized into a coherent system, can produce a material whose qualities are comparable or superior to materials composed of energy-expensive or rare constituents. Similar phenomena occur in the brain and the computer: neurons and transistors, organized appropriately, yield unexpected power. An economical and sustainable approach to technology demands that we gain a better understanding of this insight and a primary step is to provide a general mathematical framework in which to explore it. In order to produce such a framework it is necessary to survey the field and locate the simplest common themes.

Indeed there are similarities that run through all such studies. Whether working in physics, sociology, or computation, interactions are based on some sort of exchange between individuals. For example energy is transferred from particle to particle in a material, and ideas are transferred from human to human in a society. Often the unit of exchange can be described in terms of information, and in the present proposal I will take the essence of interaction to be the conveyance of information. One can think of information as that which guides, influences, offers direction to an entity.

Self-similarity also seems to be inherent in the workings of nature and certainly in the examples presented so far. From atoms to cells to humans to societies, each entity seems to be composed of an interacting set of smaller entities. But how can we say what is happening at the large scale if the small scale actors remain unexplained? We must consider entities at any level to be *black boxes* that have some known properties but unknown inner-workings. We are tasked with predicting how an organized system of these black boxes will behave. It is this whole issue that I intend to explore for this grant.

Mathematics, and category theory in particular, has the power to thematize and organize our thought on this matter, providing a language and toolset from which to work. Category theory is the language of structure. Famously, its emphasis has always been placed on the *interactions* between objects rather than on individual objects in isolation. This simple change of viewpoint is arguably its primary contribution and source of strength, tying together structure and function in an appropriate way for the first time.

Since its invention in the 1940s, category theory has revolutionized mathematics. Although sometimes denigrated as "just a convenient language" it is now becoming clear that a well-designed language is an invaluable aid to innovation. Consider for example the field of computer programming, in which choosing an appropriate and flexible programming language is crucial to the long-term success of a project. By emphasizing the principles that govern a given subject, category theory leads its practitioner to pose the right questions and phrase his or her findings within a framework that is highly self-consistent and interoperable with results obtained by other mathematicians.

A real advance in fields such as information integration require an adequate formulation of the complex relationship between structure and function in any system. Indeed these two aspects are deeply intertwined and effect one another in complex ways. Effective information management systems must account for the dynamic nature of real life, in which changing demands on function often require novel changes of structure, and changing availability of resources must be managed as seamlessly as possible. To create such systems requires that we get to the heart of the interaction between structure and function, and articulate it rigorously.

I propose to study the basic structures and dynamics of interaction using the language of category theory. One novelty in my approach is that it can account for the complex interdependencies between structure and function. Another novelty is that it will describe information exchange not at the physical level of packets and symbols (a la Shannon) but at the level of semantic meaning, which is what humans are generally speaking about when we speak of information [Sav]. In Chapter 2, I will discuss two approaches to this endeavor, which I will now summarize.[1]

The first approach is to develop a new "communication protocol" framework [2] by which two databases or agents can communicate new concepts back and forth. With this method,

[1] None of this work will result in any *environmental impacts* beyond those caused by airplane travel.

[2] this is almost completely unrelated to the classical notion of communications protocol, such as TCIP/IP

the agents continually build up a shared common ground language as well as increasing their own individual knowledge base, leading to increased communication efficiency over time. A basic "communication protocol" has already been worked out, and I will discuss it in Section 2.1. Both details and applications have yet to be pursued, and there appear to be interesting directions for extending the theory itself. Understanding interactions and information exchanged between distributed agents is crucial for fusing information.

The second approach is quite rich, and could potentially incorporate many different academic research strands from the last 70 years, including category theory, information theory, cybernetics, Bayesian probability theory and machine learning, the theory of programming languages, neuroscience, and algebraic topology. I will discuss a category-theoretic model of systems (sets of interacting parts aggregated into a single unit) and their behavior in Section 2.2.

The above research project will have several applications, which I will discuss throughout Chapter 2. The inherent excitement that tends to surround hot topics and prioritized research agendas often causes a somewhat frenzied temperament by the research community. Although such an influx of energy may yield powerful results, it also leads to a wasteful reduplication of efforts and a lack of holistic perspective. A principled and coherent approach has distinct advantages. My proposed research is dedicated to offering a mathematical foundation for, and to suggesting a simple and unified perspective on, many trending subjects. These include:

i. formulating effective communication strategies between unfamiliar agents, such as between two databases that "can't talk to each other" to facilitate and enable information integration (Section 2.1);

ii. offering a theoretical approach to the reverse engineering of various kinds of *black boxes* to gain insight (p. 14);

iii. understanding the relation between structure and function in complex systems, which can be used to specify and model examples brains, computer networks, biological systems, and economic markets (p. 17);

iv. providing a rigorous foundation for data flow diagrams and production recipes, e.g. for companies manufacturing pharmaceuticals, engineers simulating multifaceted dynamical systems, programmers designing software, and logistical supply chain planning and management (p. 17); and

v. establishing an overarching framework for problems in machine or human learning and Bayesian reasoning (p. 17).

Before explaining my research agenda for the present grant and how it applies to the above topics, I will summarize my previous work, most of which was made possible by two consecutive Office of Naval Research grants (N000140910466 and N000141010841).

1.2 Previous work

Since 2008 I have working on projects that support the above research program. For the most part my work has centered around finding category-theoretic or algebro-topological formulations for the structure and operation of databases. The results have generated a healthy amount of interest, both from within academia and from without. Below I will describe the subjects of relevant papers I have published and talks I have given. In Section 2.3, I will discuss my active collaborations as well as a transition of some of my academic research into mainstream production.

1.2.1 Papers and invited presentations

A list of papers I have published and talks I have given on the subject of categorical information theory can be found in my CV. Below I will outline a few broad topics.

Simplicial databases

A database, as formulated by Codd, is a set of relations. This mathematical formalism provided a unified language and toolset that together led to major advances in the field. However, his model has serious weaknesses. For example, databases involve interacting tables, and Codd's notion treats this fact as an afterthought. Thus the model is not fit for understanding schemas holistically, which is necessary when comparing them. My intention is to improve communication between disparate entities, and hence it is imperative that schemas be nicely comparable. To that end I developed simplicial databases,[3] which are geometric objects that capture the interrelationship between various tables in a schema.

The geometric nature of these databases has valuable consequences. For example, a simplicial schema often looks like what it is intended to model: one can model one-way airplane tickets using a line segment (1-simplex) between two points, and then simply glue this object to itself, going backwards, to model round-trip tickets. The mathematics follows suit. Also, one can draw paths through a simplicial database and information will be carried from the start point to the end point of the path. More precisely, any such path entails a query against the database in which data is entered into the table at the source of the path and resultant data is output from the table at the target of the path. This was discussed in a second paper.[4]

Several mathematicians saw the value in this approach and invited me to speak at various universities and conferences. Some high-level people in industry (including a vice president at Amgen and a director at Johnson & Johnson) also appreciated the work and I have worked closely with members of both companies.

[3] D.I. Spivak. (2009) "Simplicial databases". ePrint available http://arxiv.org/abs/0904.2012. (Supported by ONR grant N000140910466.)

[4] D.I. Spivak. (2010) "Table manipulation in simplicial databases". ePrint available http://arxiv.org/abs/1003.2682. (Supported by ONR grant N000140910466.)

Databases as categories

At some point I began to recognize that foreign keys (functional connections between tables) were essential, and I worked to make them more prominent in the theory. Realizing that even data columns could be considered as foreign key columns allowed me to say that a database schema *is* a category $\mathcal{C}$, and that an instance *is* a functor $I\colon \mathcal{C} \to \mathbf{Set}$. This greatly simplified matters, and the idea has been very fruitful.

For one thing, it allowed a good definition of schema integration and data migration. I published a paper on this in 2012.[5] For another, taking the Grothendieck construction of an instance converts any relational database (collection of tables) into an RDF triple store, a common format for storing semi-structured data. Formulating functorial connections between the viewpoints of multiple users in "the cloud" will create an atlas of knowledge, so that information can be more coherently shared between them. Alignment of concepts can be guaranteed without requiring a unified naming convention. As an added benefit, this perspective allows one to view queries and constraints in terms of lifting problems, in the sense of homotopy theory, so that theorems from algebraic topology can be applied to make sense of data. I submitted a paper on this subject in 2012 as well.[6]

The simplicity of the categorical approach leads to flexibility in the sense that it is easy to extend the theory. For example, I was able to use monads in much the same way they are used in programming language theory to relax the atomicity requirement for data in a database. Using the Kleisli construction allows lists, sets, probability distributions, accuracy estimates, and other types of additional information to occur in each cell of a database. A paper on this subject is complete and will be submitted to the PODS database conference this year.[7]

Ologs as outreach

Improving communication between related entities is a core motivation of my research, and as such I felt it was important to reach out to other disciplines in order to improve communication between our respective academic fields. To that end I am teaching a course at MIT in Spring 2013 called *Category Theory for Scientists*, where I will discuss the basic tenets of category theory, including functors, natural transformations, adjoints, monads, and operads. But I will do so in a very down-to-earth way that includes real-world examples.

The same motivation led me to put a linguistic twist on categories, so that they could be drawn as human readable box-and-arrow diagrams. I called these structures *ologs* and wrote a paper on the subject that was published in 2012.[8] I have received a good deal of appreciation for that paper.

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[5] D.I. Spivak. (2012) "Functorial data migration". *Information and Computation*. (Supported by ONR grant N000141010841.)

[6] D.I. Spivak (2012) "Database queries and constraints via lifting problems". Submitted to *Mathematical structures in computer science*. (Supported by ONR grant N000141010841.)

[7] D.I. Spivak. (2012) "Kleisli database instances". ePrint available, http://arxiv.org/abs/1209.1011. (Supported by ONR grant N000141010841.)

[8] D.I. Spivak, R.E. Kent. (2012) "Ologs: a category-theoretic foundation for knowledge representation". *PLoS ONE*. (Supported by ONR grant N000141010841.)

The olog concept also led to an unexpected collaboration with a distinguished MIT professor in materials science, named Markus Buehler. Together we have published four papers in the last 2 years.[9][10][11][12]

[9]D.I. Spivak, T. Giesa, E. Wood, M.J. Buehler. (2011) "category-theoretic analysis of hierarchical protein materials and social networks". *PLoS ONE*. (Supported by ONR grant N000141010841.)

[10]T. Giesa, D.I. Spivak, M.J. Buehler. (2011) "Reoccurring patterns in hierarchical protein materials and music: the power of analogies". *BioNanoScience*.

[11]T. Giesa, D.I. Spivak, M.J. Buehler. (2012) "Category theory based solution for the building block replacement problem in materials design." *Advanced Engineering Materials*. (Supported by ONR grant N000141010841.)

[12]J.Y. Wong, J. McDonald, M. Taylor-Pinney, D.I. Spivak, D.L. Kaplan, M.J. Buehler. (2012) "Materials by design: merging proteins and music." *Nano Today*.

Chapter 2

Research effort

Category theory is an excellent language for describing structure and function of any kind. Past work, such as [Ros] and [DK], employed category theory and was quite influential, yet it used categories at a somewhat straightforward and superficial level. On the other hand the articifical inteligence and database communities have formulated the disjoint concept of structural and functional views for knowledge representation [BFL]. Real advances in a given subject can emerge when we investigate its basic foundational aspects and articulate new findings in an appropriately expressive language. To do so requires a deep intimacy and facility with that language, and this seems to be lacking in prior work. The language of modern category theory, including operads and higher-categories, is well-suited to making significant progress in the field of distributed intelligence and information fusion because these advanced mathematical concepts can deal with functions and structures in the same framework.

My research involves exploring the complex nature of information and communication, in terms of both their structure and their function, and formulating the results in the most appropriate mathematical language. Some of this work is similar in intent to that of [BRO], modeling the aggregate properties that result from information sharing in complex systems. However, that work is more closely tied with Shannon's Information Theory and probability rather than category theory. As such it is useful for low-level calculation but does not adequately enable transition between the multiple user-defined hierarchical levels, which are ubiquitous in the field of information management. The category theoretic description successfully navigates such transitions. It also has the added feature of allowing us to create and rigorously compare different models of the same phenomena, using functors.

In the following sections I will describe two different applications of category theory to the study of interaction. In Section 2.1, I will describe a framework with which to model networks of interacting agents that learn from each other. In Section 2.2, I will describe how interacting entities at one level can come together to form a single higher-level entity. In particular, I will show how recursive functions, such as the factorial function, can be specified in this model, thus offering a new method for integrating databases and programming languages.

For more on category theory and its use in computer science or general modeling, see [Mac], [BW], and [SK].

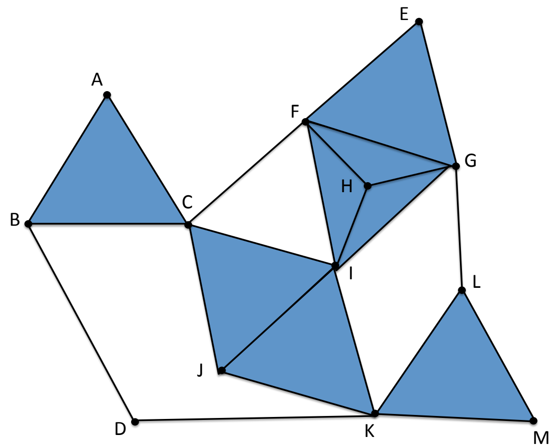


Figure 2.1.1: A simplicial complex modeling a network of interaction. The vertices correspond to agents and the higher simplices correspond to interactions. For example there is a 2-way interaction between agents B and D and a 4-way interaction between agents F, G, H, and I.

2.1 Networks and communication

Information is never confined to a single agent. That which is confined to a single agent is akin to hardware, whereas information behaves more like software. Instead, agents constantly exchange information with other agents, so as to align their behaviors (reducing the inefficiency arising from chaotic misalignment of behaviors) and to collect a set of effective strategies for survival in the current environment. In particular alignment of ontologies is an important research area in information fusion and data management. However, researchers in ontologies alignment have not addressed dynamic information.

For a large number of scientific, industrial, and militaristic pursuits, it is important to understand how learning is transferred and how information travels through a network of such interacting agents. Although I intend to eventually study systems whose connection patterns evolve over time, as a first pass I will consider a network of interacting agents that is fixed in its pattern of connection.

I model this as a simplicial complex X in which the vertices correspond to agents and the n-simplices correspond to $(n+1)$-ary interactions (see Figure 2.1.1 at the top of page 9). Each simplex $x \in X_n$ is assigned a database schema $\mathcal{C}(x)$.[1] For various purposes we could assume that these schemas were identical to one another ($\mathcal{C}(x) = \mathcal{C}(y)$ for all $x, y \in X$), or that the schemas are possibly different but do not vary in time, or that the schemas are allowed to vary in time. Each simplex is also assigned a database instance on $\mathcal{C}(x)$ which does vary in time as interactions occur. The application of Sowa's Information Flow Framework to this model was described in [SK, Section 4].

I will now discuss the "communication protocol" I developed with Mathieu Anel in which we show how one agent can learn new ideas from another. Grounded in this protocol,

[1] See [Spi] for formal definitions of database schemas and mappings between them.

one can study effective communication strategies between unfamiliar agents. To be precise, suppose that K_A and K_B are databases corresponding to the knowledge of agents A and B, and let K_{AB} denote their current common ground, i.e what is mutually understood by both. We have a diagram

$$\begin{array}{ccc} K_{AB} & \xrightarrow{i_B} & K_B \\ {\scriptstyle i_A}\downarrow & & \\ K_A & & \end{array} \tag{2.1.2}$$

which says that everything in the common ground K_{AB} has an interpretation in A's knowledge K_A, given by i_A (and similarly it has an interpretation in K_B, given by i_B).

Suppose that A wants to communicate some new concept to B. She expresses this concept in terms of concepts that are already familiar to both, i.e. concepts within the common ground. For example, to explain what a *supernova* is, she might use terms like *star* and *explosion* from the common ground. We formulate such a communication attempt as a diagram

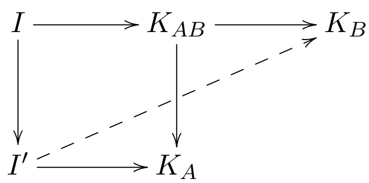
$$\begin{array}{ccccc} I & \longrightarrow & K_{AB} & \xrightarrow{i_B} & K_B \\ \downarrow & & {\scriptstyle i_A}\downarrow & & \\ I' & \longrightarrow & K_A & & \end{array} \tag{2.1.3}$$

In other words I is a mini-schema corresponding to certain shared concepts (e.g. star, explosion) and their relationships within the common ground K_{AB}, and I' is an extension of it that includes the new concepts from K_A (e.g. supernova) as an extension of the shared concepts.

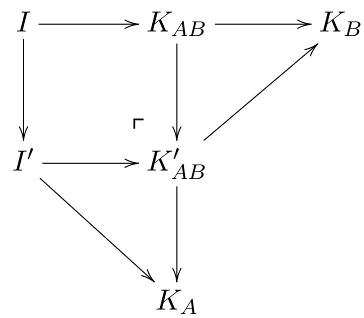
Given a communication attempt by agent A as in Diagram (2.1.3), agent B can do one of three things: he can interpret the new ideas within his existing knowledge, he can learn the new ideas by adding them to his knowledge, or he can reject the new ideas altogether. In the first two cases, the common ground is updated to include the new ideas; in the last case everything remains unchanged by the communication attempt. We now describe these three possibilities using category theory.

Interpret If B can find a lift $I' \to K_B$, we say that B has *interpreted* the information. Then the common ground K_{AB} can be updated by pushing out the diagram $I' \leftarrow I \to K_{AB}$.

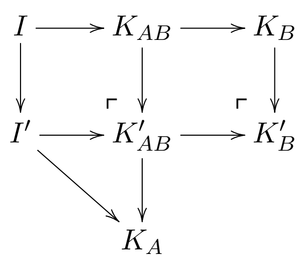
If B can interpret the new information,



he pushes out the common ground.



Learn If B cannot find a lift, he can still learn the new information (e.g. simply add a concept of supernova to his schema and to the common ground). To do so, he pushes out both K_{AB} and K_B along $I \to I'$. This ensures that his new understanding of supernova is appropriately related to his existing understanding of atom and explosion.



Reject If B chooses, he can disregard this communication attempt entirely and neither interpret it nor learn from it, leaving the initial setup as in Diagram 2.1.2

Regardless of B's choice, the communication attempt begins and ends in the same setup: two knowledge databases connected by a common ground, as in Diagram (2.1.2). In the case of interpretation, the common ground is updated, but B's knowledge remains the same; in the case of learning, both the common ground and B's knowledge are updated; in the case of rejection, everything stays the same. If A and B are two domains of knowledge then these three possibilities should be relevant to the problems in transfer learning:

i. when can learning transfer between A and B?

ii. what are the available mechanisms of transfer?

It is plausible that Homotopy Type Theory ([Awo]) may be useful here. Instead of considering 1-categorical lifts and pushouts, we might instead use n-categorical versions. This way, we could create an equivalence between terminologies without forcing equality; the equivalences themselves become an important part of the data. One may then leverage homotopy type theory in a deep way to study the above interpret–learn–reject model and perhaps offer a more interesting interpretation of learning.

Mathieu Anel and I have verified that this "communication protocol" works as it should for databases, when the communication attempts are guided by experts. It would be interesting to study how such a communication process could be specified and mechanized. This may lead to a procedure for two databases (or more generally two agents) that initially have only a small amount of shared information to build rapport, i.e. to automatically improve their ability to efficiently exchange information. This ability to converse with mathematical precision, building shared knowledge and thereby increasing efficiency, will provide a much more effective approach to information integration than what currently exists.

2.2 Emergence of aggregate properties from small-scale interactions

There is a ubiquitous phenomenon in the natural world, whereby multiple entities at one level come together to form a single entity at a higher level. Generally, it seems that the subordinate entities are not all exposed to the outside world, some are interior to the larger entity. For example, the human body has many nerve endings on the skin and in the muscles—these exchange information with the outside world; however, most nerve cells in the body are interior in that they only receive information from, and transmit information to, other nerve cells. Similarly a single military operation requires and is composed of a complex interaction between myriad personnel, only some of which are exposed to violent combat.

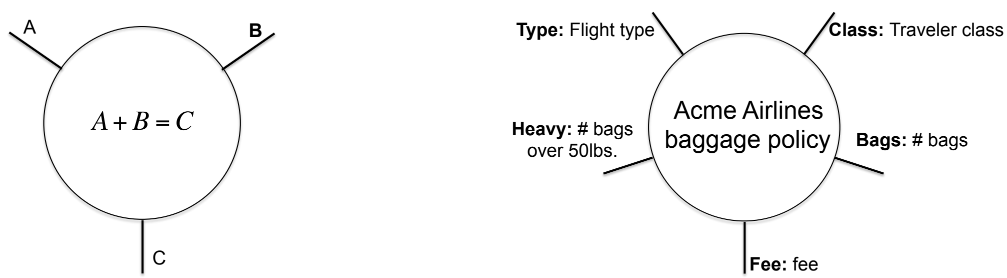
The term *holon* is sometimes used ([Cal], [LCM]) to refer to anything which is both a whole and a part, and the concept is important here. A holon is comprised of subordinate holons, it interacts with holons of its own level, and systematic interaction at this level serves to form a holon at a higher level. A molecule is a holon in that it is comprised of atoms and interacts with other molecules to form materials; a human is a holon in that is comprised of organs and interacts with other humans to form societies. In what follows I will propose a mathematical model for holons, their interactions, and the formation of larger holons from smaller.

An adequate model of holons must take the notion of self-similarity seriously. For this purpose I will employ the theory of operads from algebraic topology. Operads and their algebras are used to express the idea summed up in the motto *e pluribus unum*: Out of many, one. Thus it is precisely what we need to model holons. For more on operads see [Lei, Chapter 2]. In what follows I will mostly opt to use the more commonly-known term *entity* rather than holon, but I mean them synonymously throughout.

Even when we do not understand the inner workings of an entity, we can still represent it in terms of the ways by which it interacts with the outside world. Thus we model each entity as having some number of information channels that carry data in a given range, such as numbers, pitches, how hot something is, how beautiful something is, etc. We do not demand that these channels carry simple boolean variables; they may carry any variable that can be subjectively measured by the entity.

Each entity maintains a relationship between the data on its channels, in the sense that it constantly regulates or conforms to various parameters in its environment. This relationship is the very identity of the entity. For example, each human maintains a relationship between what he sees, what he hears, what he feels, and what he does. Such mechanisms are found in the biological tendency toward homeostasis, as well as in the seemingly dissimilar example in which force is transferred through a material. Indeed, we might say that a material maintains a relationship between the forces put upon it. Whether we are speaking in terms of biology or physics, entities are identified by the way they act, and this way of acting can be interpreted as a relationship between a certain set of phenomena. When the identifying relationship cannot be maintained the entity loses integrity, and often it dies or breaks.

The discussion can be brought down to earth by some examples. Figure 2.2.1 provides a mathematical example and a corporate example. Hopefully the similarity is clear: Just as a ticket agent is tasked with maintaining a relationship between various travel and baggage parameters, so is the mathematical concept of *plus* tasked with maintaining a relationship between three integers.



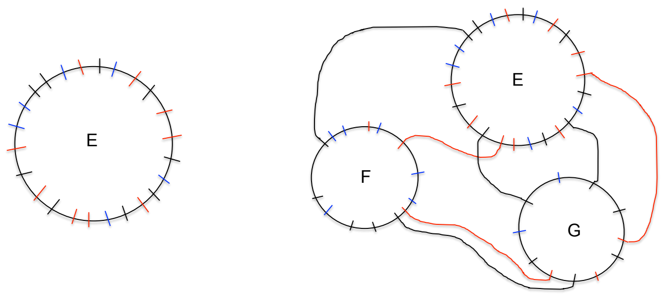
$A + B = C$		
A	**B**	**C**
0	0	0
0	1	1
0	-1	-1
1	0	1
-1	0	-1
1	1	2
⋮	⋮	⋮
11	-17	-6
⋮	⋮	⋮

Acme Airlines baggage policy				
Type	**Class**	**Bags**	**Heavy**	**Fee**
Int'l	Econ	1	0	$25
Int'l	Econ	1	1	$25
Int'l	Econ	2	0	$50
Int'l	Econ	2	1	$50
Int'l	Econ	2	2	$100
Int'l	1st	0	0	$0
⋮	⋮	⋮	⋮	⋮
Dom	Econ	2	2	$150
Dom	1st	0	0	$0
⋮	⋮	⋮	⋮	⋮

Figure 2.2.1: Two relational entities, one mathematical the other corporate. They are black boxes, drawn as circles with lines emanating from them, each line of which represents an information channel. Below each relational entity is a table representing the relationship it maintains.

We draw each entity E as a labeled circle with lines (or *cables*) emanating from it; each cable represents an information channel on which the entity receives and transmits data. Each channel is tasked with carrying a prescribed set of possible values. The entity maintains a certain relationship between all such information, and this relationship can be captured by a table whose columns are the channels and whose rows represent all the legal states of the entity. Mathematically, if the entity E has information channels carrying sets $A_1, A_2, \ldots, A_n$ then the table represents a subset $R_E \subseteq A_1 \times A_2 \times \cdots \times A_n$, and it is this n-ary mathematical *relation* that we have informally been referring to as the relationship maintained by E.

Entities can of course have many channels. On the left below is an entity with 24 channels of a few different types. On the right we see three different entities interacting along various channels. At any given time each channel is carrying one value, so when entities share a channel it means that they simultaneously perceive whatever value is currently on that channel.



An entity may be known as a system of smaller entities, or its inner workings may remain a mystery. In the case of the Acme Airline baggage policy in Figure 2.2.1, one could look at the table and try to reverse-engineer the organization that produced it. This may amount to guessing the broad outline of the pricing structure or, if that is known, guessing the price values associated with the components of that structure.

Figure 2.2.2 below shows a component structure that would indeed give rise to the Acme Airlines baggage policy table from Figure 2.2.1. Note that one of the interior holons is in fact the *plus* holon whose table of values was given in Figure 2.2.1.

The mathematical theory of operads (i.e. multicategories) and their algebras is perfectly suited for formalizing this whole idea. In general it allows us to produce a relation on a whole entity by wiring together relations on subordinates. The picture below is reminiscent of the structure of a brain. The idea is that the stimulus-response pattern of the whole entity E is dictated by those of its parts, E_1, E_2, and E_3.

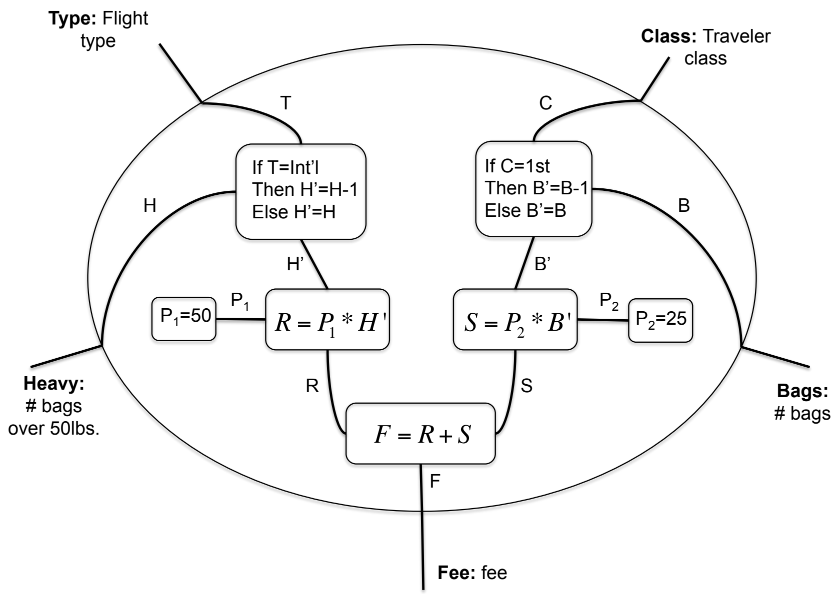
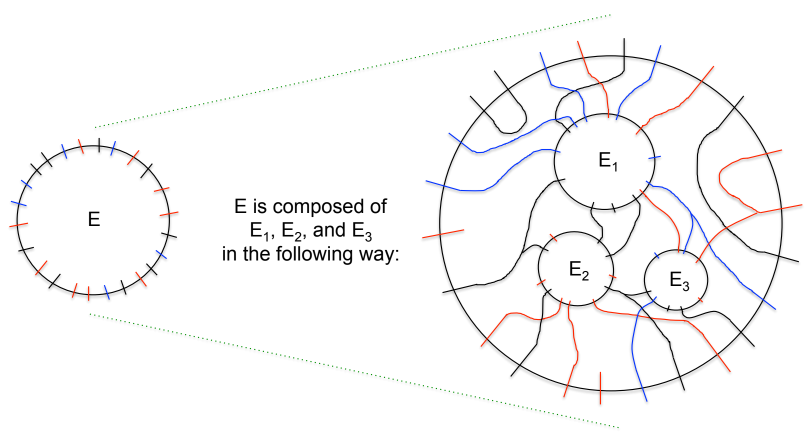


Figure 2.2.2: The Acme Airlines baggage policy holon (Figure 2.2.1) is built up out of seven sub-holons. One might zoom in to see the internal structure within each of these smaller holons or take them as black boxes.



(2.2.3)

Whatever the brain is, we know something about its structure and something about its function. We know it includes a set of interconnected neurons (with all sorts of additional complexity, e.g. glial cells), the connection pattern of which possibly changes over time; and we know that the brain can react to stimuli. The connection matrix in the brain, called the connectome [STK], may prove as important for understanding intelligence as the genome is for understanding basic biological life.

There is a clear commonality between the above operadic model and the known structure of the brain (though the model is significantly simplified), and I believe that the above model is sufficiently general to capture functionality such as stimulus response. For example cognitive priming, which can be roughly described as ripples of activation spreading between various areas of the brain [Kah, p. 53], is easily formulated in an operadic model. The creation of higher-level meaning emerges from the ability, provided by our model, to group together and analyze the behavior of various interacting sets of neurons that form specialized modules at any level. This research will yield insights into our knowledge gaps between neuroscience and cognitive science.

However, the brain is capable of seemingly more advanced functionality than simple stimulus-response. In particular, we are able to calculate answers to difficult combinatorial problems and perform recursive procedures. For example the *factorial* function (Factorial(5)=5*4*3*2*1=120) is a recursive procedure that the brain is capable of handling, at least for small input values. Any viable theory of the relationship between structure and function in the brain should show how this kind of performance can manifest from structures akin to the neuronal connection pattern. The following diagram (Figure 2.2.4) is rigorously defined in the theory of operads and their algebras, and does indeed explicitly show that our proposed type of interaction is rich enough to encode recursive algorithms. In fact I have sketched a proof that any recursive algorithm in the sense of [BJ] can be encoded in this way. Thus the model may have direct application to the theory of programming languages and serve to better integrate databases and programs.

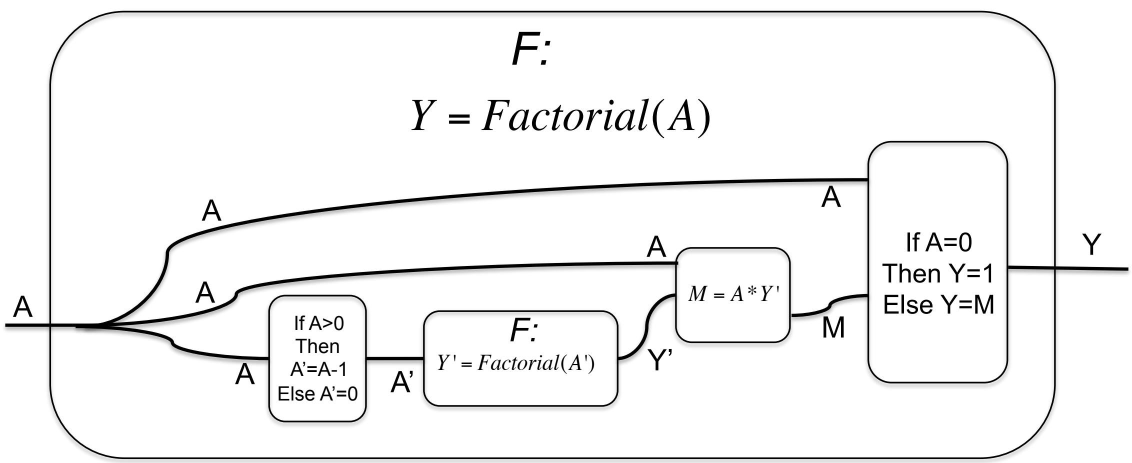


Figure 2.2.4: The factorial relational entity, defined recursively. This picture represents an equation in an operad algebra, specifying a free variable relation (F) in terms of itself and some constants, akin to the equation $x = x^2 - 3x - 5$ in $\mathbb{Z}[x]$. Just as $x = -1$ is a solution in the polynomial example, the factorial relation is the only nontrivial solution to the diagram above.

A third function of the human brain, aside from reacting to stimulus and performing multi-step computation, is the ability to model the future and predict certain things about it. We constantly anticipate upcoming events, designing and pre-configuring responses to our expected contingencies. For example, this occurs when a person plans what he or she will say on the phone, or when a business decides on a game plan to strategically position itself in next year's market. Such behavior is ubiquitous in human society. It points to the existence of an inner model of reality, which includes oneself as principle actor, on which one can pose various simulation problems and optimize results. It remains to be seen whether the operadic formalism can express and appropriately handle such issues.

The above considerations about the brain also hold in any other context of information integration or distributed artificial intelligence. The word *neuron* can be replaced by the word *agent*, and the word *brain* can be replaced by the word *system*. The point is that this model is sufficiently rich to capture much of the behavior of complex system by treating structure and function in a unified manner. It offers a convenient yet rigorous way to model agents and information exchange.

The above operadic approach is new, and it can be made precise. I have done this work already, including verifying that the axioms for a (colored) operad hold and that relations form an algebra on this operad. However, there is much to be done, as the following four paragraphs should give some sense.

Applying the operadic model to investigate complex systems like the brain or an economic market is certainly an interesting line of thought and a constant inspiration to my work. Such research has far-ranging applications, including to the simulations used in process planning or engineering design such as Simulink, where the relational entities are often partial differential equations ([Kar]), or to new NoSQL style database system, such as MongoDB, in which the structure of JSON-like documents seems to align with the operadic approach ([CD]). It may even have application to formalizing spreadsheets, which are surprisingly complex objects because they are so unstructured. Having a few principles that can describe many of the world's phenomena is inherently valuable. My specialty being mathematics, I will speak with experts in such other disciplines and continually drive toward a pure mathematical conception that captures as much of the landscape as can be done elegantly.

Another direction to explore is to use the operadic model to compare data and computation. I gave an example in Figure 2.2.4, which suggested that recursive computations can be modeled by the operadic framework (and indeed, any recursive function can be modeled this way), but I have not completely characterized the recursive functions in this setting. I am working with Nat Stapleton at MIT to understand exactly which relational entities are in fact computations. There is a long-standing divide between data and program in the field of computer science, and I am looking into providing a precise comparison between the two by situating them both within a common context.

Along with pursuing the above applications, one could also extend the mathematics itself in multiple directions. Instead of using relations (e.g. $R \subseteq A \times B \times C$), one can use probability distributions (e.g. $p\colon A \times B \times C \to [0,1]$ such that $\sum_{(a,b,c)} p(a,b,c) = 1$.) This is another algebra on the same operad, and the two algebras are comparable by a morphism of algebras (the support of a probability distribution is a relation). However, the distribution version is much richer and is related to machine learning and Bayesian reasoning. In other words, I believe that much of these subjects can be formulated as the inference of probability distributions on subordinate black boxes given probability distributions on the whole entity. I plan to work with Tom LaGatta to make these ideas rigorous and see if we can use them to advance the field.

Most interesting and pressing to me is to incorporate time, in particular change of network formation and one-directional signals, into my approach. Interaction is not a static thing. Moreover, communication channels are always noisy so data does is not perfectly transmitted, as it is assumed to be in the above model, but instead is prone to error. A first step would be to replace the above set-theoretic operad with a topological operad, so that relational entities can move in a configuration space. Here information-theoretic notions of noise and entropy could be incorporated in terms of the lengths of connection cables.

2.3 Collaborative efforts and dissemination of research

In recent years I've established contacts with people from various communities. This includes active collaboration with academic researchers in math, computer science, and materials science, as well as with directors and technicians in industrial companies such as Amgen and Microsoft.

Math faculty and postdocs

I spoke about my work on categorical databases at Johns Hopkins University in late 2011. This talk was enthusiastically received by professor Jack Morava, a well-known algebraic topologist. He wrote a paper called *Theories of anything* that connects my work on databases to classical concepts in representation theory and Thom's catastrophe theory. Morava explains that symmetry breaking in physics, chemistry, and throughout science, is on one hand a common generator of scientific data (e.g. the melting point of various substances comes from symmetry breaking). On another hand, it induces a category, i.e. a database schema, on which such scientific data naturally lives. This is a profound and potentially very powerful idea that should be explored.

My work on ologs and functorial data migration also inspired Scott Morrison (an ARC research fellow and senior lecturer at the Mathematical Sciences Institute of the Australian National University) to write a code base for a category-theoretic database engine. The current incarnation of this work can be found at `http://www.categoricaldata.net`.

I am involved in collaborations (at various stages) with various postdocs and research fellows in mathematics, including David Gepner, Nat Stapleton, Henrik Forssell, and Mathieu Anel.

Non-math professors

I am working with Adam Chlipala, an expert in the mathematical proof assistant called Coq. We are co-advising an excellent MIT undergraduate student named Jason Gross, who is implementing my categorical database ideas in the Coq proof assistant. A code base is available online: `https://bitbucket.org/JasonGross/catdb`.

I have also had a productive collaboration (four papers) with Markus Buehler's lab. Buehler works on materials science, for example investigating the structural relationships that give hierarchical protein materials such as spider silk their astounding functional properties. He was interested in collaborating with me because he needs to accurately describe materials, both their structure and their function, and this cannot be adequately done in English prose. Ologs allow for extremely precise descriptions. In an effort to capture the hierarchies his lab was seeing, I invented hierarchical ologs, which I hope to write about in mathematical detail in the near future.

Industry

It is important to me that my work relate to real informatic issues in the world. If I am studying information and communication, then my success is measured by whether people with experience in those fields find my work useful. Much of my research has grown out of conversations with Peter Gates (Johnson & Johnson) and Dave Balaban (Amgen, AMS Fellow). For example, they have gotten across to me the importance of aggregate functions. I have also been invited three times for 2-day sessions at Amgen, during which we discussed my research. These meetings made clear to me the importance of ETL, which later became a major selling point for my work on functorial data migration.

2.3.1 Transition to industry

In each meeting with the informatics group at Amgen, we were joined by Allen Brown of Microsoft. He found the olog concept to be intuitive and useful, and in fact has begun to implement ologs in the upcoming release of the Microsoft Semantic Storage System (SSS), which is expected to be used by Amgen. He also considers the monadic database instances work to be "dynamite" and is employing it also in the SSS.

I am quite pleased with this transition from academic research to industry-level software. I take it as evidence that my research is grounded in reality, as opposed to being stuck in the ivory tower. Interestingly, many of my colleagues at MIT still consider what I am doing to be pure math because it adheres to the rigor and aesthetic that pure math demands. My expertise is in listening to what is being done and what is needed in the real world and in turn, offering appropriate structures and tools that are well-known to be effective in mathematical research. It is gratifying that these same tools are found to be useful by industry.

I hope and believe that this will be the first of many instances in which my work is transitioned from academia into practice.

2.3.2 Transition ideas to DOD and Air Force relevance

This project offers new high-level approaches to information integration and distributed artificial intelligence. Today's algorithms tend to work from the bottom up, analyzing big data for known patterns. This can be useful but does not provide adequate opportunity to employ the domain knowledge of human experts. Human intuition and insight relies on the ability to relate various levels of organizational complexity, and this is not part of the current approach to big data. In this proposal I have clearly outlined a new method for the transfer of learning between individuals by prescribing a "communication protocol" and a new method for the extraction of large-scale knowledge from the interaction pattern of small-scale data. It is clear that analyzing an essay in terms of its word count or the author's vocabulary size is vastly different than extracting the its theme or the author's purpose (although both analyses can be useful). Just so, the sort of summary yielded by the typical bottom-up approach to big data is of a different nature than that yielded by my approach. It seems clear that in order to make an intelligent whole out of a set of distributed agents we must weave their contributions together to form a larger narrative rather than simply reporting low-level patterns and trends.

I look forward to collaborating with research staff of the Air Force Research Lab, to transition ideas from basic research to applied research.

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Chapter 3

Grant Logistics

3.1 Principle investigator (PI) time

The PI will devote approximately 9 calendar months per year to this project. Graduate students, serving as Research Assistants (RAs), will devote 5.5 to 7.6 months per year to this project. They will perform academic research on various mathematical problems that arise, as well as suggesting and working out extensions to the theory. The PI will be responsible for all employed graduate students.

I am planning to submit a proposal to ONR to renew my current grant that will expire on May 31, 2013. This grant will focus on monadic and hierarchical databases and their fundamental structure as well as operations.

3.2 Facilities

MIT provides basic office space and library services for faculty.

The campus at the Massachusetts Institute of Technology is networked by both a wired 100/1000Mbps Ethernet LAN and campus-wide 802.11b/g wireless networks. MIT utilizes both proprietary and open source workstations and servers, including Linux, Unix and Macintosh platforms. In addition to office workstations, MIT provides clusters of workstations throughout campus running a customized distribution of Linux derived from Red Hat Enterprise 4. Network security is provided by the Kerberos authentication protocol.

The Math Department maintains its own subnet and domain, and provides separate email, file storage, computational, and internet services for faculty and staff over a Gigabit Ethernet LAN. Faculty workstations run Fedora Linux, and department servers run Red Hat Enterprise Linux and Fedora Linux. The department maintains a variety of network printing services are available for networked computers.

3.3 Current Project and Pending Proposals

I have an existing grant from the ONR, detailed below; it is set to end 2013/06/01. I am currently applying for a 3-year grant from ONR.

3.4 Special test equipment

None.

3.5 Equipment

None.

3.6 High performance computing availability

Not needed.