



Mapping spaces in quasi-categories

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Homotopy everything

- Spaces as ∞ -groupoids
- Homotopy coherence
- Don't want to choose composition law.



Solution: Weak Kan complexes

- Boardman and Vogt: “Homotopy invariant algebraic structures on Topological spaces”
- Solution: Use simplicial sets as categories:
 - 0-simplices are called objects;
 - 1-simplices are called arrows;
 - Higher simplices are “higher composition relations.”
- Weak Kan condition:

• For $0 < k < n$, right lifting property

$$\begin{array}{ccc} \Lambda_k^n & \xrightarrow{\forall} & X \\ \downarrow & \nearrow \exists & \uparrow \\ \Delta^n & & \end{array}$$

- If C is a category then $N(C) \in \mathbf{sSet}$ satisfies this property.
- Simplicial sets X with this structure are called “Quasi-categories.”



The model structure

- Joyal found model category structure on **sSet**:
 - The fibrant objects are the quasi-categories.
 - Weak equivalences generalize equivalences of categories.
 - Denote it **sSet_J** “Joyal”; distinguish from **sSet_K** “Kan”.
- Lurie proved there exists a Quillen equivalence

$$\mathcal{C}: \mathbf{sSet}_J \rightarrow \mathbf{sCat}$$

(Bergner’s model structure on **sCat**).



Mapping spaces

- Lurie defines several versions of mapping spaces for $a, b \in Y$:
 - $\mathrm{Hom}_Y^R(a, b)$,
 - $\mathrm{Hom}_Y(a, b)$,
 - $\mathrm{Hom}_Y^L(a, b)$, and
 - $\mathcal{C}(Y)(a, b)$.
- Of these, only $\mathcal{C}(Y)(a, b)$ has composition law.
- \mathcal{C} model is difficult to work with (as we will see).
- Lurie uses “Straightening and Unstraightening” to prove Quillen equivalence — somewhat unenlightening.



Work with Dan Dugger

- Obtain several equivalent versions of composable mapping spaces.
- Understand Lurie's $\mathrm{Hom}_Y^R(a, b)$, $\mathrm{Hom}_Y^L(a, b)$, and $\mathrm{Hom}_Y(a, b)$:
Dwyer-Kan “homotopy function complexes.”
- Use these ideas to prove Lurie’s result in new way.
- Redevelop the Joyal model structure (to keep work self-contained).



Definitions

- A necklace is a simplicial set of the form

$$T = \Delta^{n_1} \vee \Delta^{n_2} \vee \dots \vee \Delta^{n_k}.$$

- Vertices V_T , Joints J_T , endpoints α_T, ω_T
- Simplexification $\Delta[T]$.
- Morphisms: maps of simplicial sets preserving endpoints.
- Category: $\mathcal{Nec} \subset \mathbf{sSet}_{*,*}$.



A few notes on necklaces

$$\mathcal{T} = \Delta^{n_1} \vee \Delta^{n_2} \vee \dots \vee \Delta^{n_k}.$$

- Note that every simplex Δ^n is a necklace.
- Note that one can concatenate necklaces $T \vee U$.
- Given $X \in \mathbf{sSet}$ and $a, b \in X_0$, understand $(\mathcal{N}ec \downarrow X_{a,b})$.
 - $T \rightarrow X$ such that $\alpha_T \mapsto a$ and $\omega_T \mapsto b$.



Two models of \mathfrak{C}

$X \in \mathbf{sSet}$, $a, b \in X_0$.

- Define $\mathfrak{C}(X)(a, b) = (\operatorname{colim}_{\Delta^n \rightarrow X} \mathfrak{C}(\Delta^n))(a, b)$.
- For $T \in \mathcal{Nec}$ it is easy:
 - $\mathfrak{C}(T)(\alpha_T, \omega_T)$ is a cube $(\Delta^1)^{|V_T - J_T|}$.
- Proposition: $\mathfrak{C}(X)(a, b) \cong \operatorname{colim}_{T \rightarrow X_{a,b}} \mathfrak{C}(T)(\alpha_T, \omega_T)$.
 - More convenient: colimit of simplicial sets.
 - Have several “isomorphic” formulas for simplices in $\mathfrak{C}(X)(a, b)$.
 - Still not homotopical.
- Nerves are nice homotopically: Quillen’s theorem A.
- Define $\mathfrak{C}^{nec}(X)(a, b) = N(\mathcal{Nec} \downarrow X_{a,b})$
- Theorem: $\mathfrak{C}(X)(a, b) \simeq \mathfrak{C}^{nec}(X)(a, b)$.
- Note that the mapping spaces in $\mathfrak{C}^{nec}(X)$ are composable.



Homotopy function complex

Dwyer and Kan (1980): Homotopy function complexes.

- In any model category \mathcal{M} , define $\mathrm{hMap}_{\mathcal{M}}(X, Y) \in \mathbf{sSet}$ in several ways:
 - Cosimplicial resolutions: $\mathcal{M}(Q^{\bullet}, \hat{Y})$.
 - Simplicial resolutions $\mathcal{M}(\tilde{X}, R^{\bullet})$.
 - Nerves of certain zig-zag categories.
- Our case $\mathcal{M} = \mathbf{sSet}_{*,*}$.
- Various cosimplicial resolutions of Δ^1 as Q^{\bullet} equate $\mathrm{hMap}_{\mathcal{M}}(\Delta^1, X_{a,b})$ with
 - $\mathrm{Hom}_X^R(a, b)$,
 - $\mathrm{Hom}_X^L(a, b)$,
 - $\mathrm{Hom}_X(a, b)$; and more.



The Quillen equivalence

Using these techniques, we can prove

- Proposition: \mathcal{C} preserves products: $\mathcal{C}(X \times Y) \simeq \mathcal{C}(X) \times \mathcal{C}(Y)$.
- Proposition: The counit $\eta_D: \mathcal{C}ND \rightarrow D$ is a weak equivalence for all fibrant $D \in \mathbf{sCat}$.
- Theorem: $X \rightarrow Y$ is weak equivalence in \mathbf{sSet}_J if and only if $\mathcal{C}(X) \rightarrow \mathcal{C}(Y)$ is weak equivalence in \mathbf{sCat} .
- Corollary: The adjunction $\mathcal{C}: \mathbf{sSet}_J \rightleftarrows \mathbf{sCat} : N$ is a Quillen equivalence.