

COMMUNICATION NETWORKS
MIT LINGUISTICS: SEMANTICS READING GROUP, 2010/09/15

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ABSTRACT. Every person has his or her own idiolect, or way of using language; in fact every subculture has its own language. I propose that every interacting group, however small, has its own language. Two people engaged in conversation have a common ground, which each can use to explain new concepts to the other. As they learn from each other, the common ground grows with shared terminology and experience, and the ability to communicate new concepts is enhanced.

Consider then the network of all human interactions. These interactions are not all “2-way”: when three people have a group conversation it brings a different dynamic than when only the pairs can privately converse. I’ll explain how the geometric notion of “simplicial complex” can be used to model a network of n -way interactions.

Thus every node and every n -way connection in a network has its own language and worldview. To make this precise we must define what these worldviews are. To that end, I’ll explain what a category is and give several examples. Then I’ll propose that we can model worldviews as categories. The network of human interactions becomes a “sheaf” of categories on a simplicial complex: local languages interacting to create a higher-order entity trying to understand its world.

I. Introduction

A. Main idea

1. A network of human interaction
2. Trying to understand our world
 - a. Given perceptions
 - b. Persist by classifying them so as to recognize patterns
3. Networks on many levels: human to neuron
4. What is the language of a network?

B. Category theory – the language of mathematics.

1. History
2. Philosophically
3. Value – layering abstraction
4. Ubiquity in math

C. Simplicial complexes

1. These will be the shape of our network
2. What is a simplicial complex? See handout Section ??.

II. What are categories and functors?

A. Categories

1. Definition of category
 - a. Basic idea: nodes, arrows, equivalences of paths.
 - b. See handout Section ??.
2. Examples
 - a. A set
 - b. The category of sets
 - c. A graph
 - d. The category of graphs
 - e. The open sets of a space
 - f. The simplices of a simplicial complex.

B. Functors

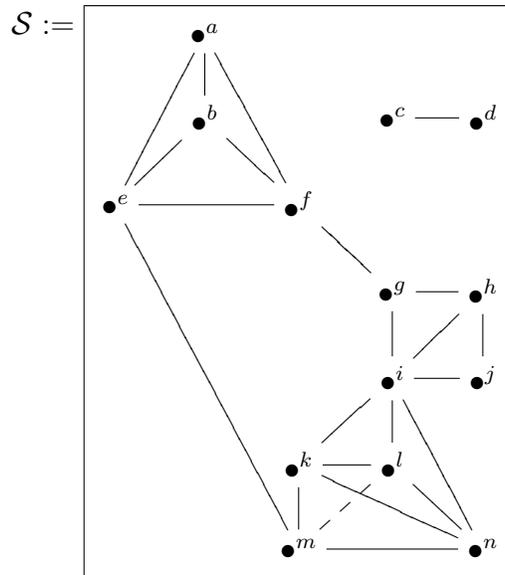
1. Definition of functor. See handout Section ??.
2. Examples
3. Sheaves: if $V \subset U$ then given something throughout U I get something throughout V .
 - a. Temperature functions

- b. Laws
 - 4. The category of categories.
 - III. A linguistic look
 - A. Categories as worldviews
 - 1. Ontology as worldview. See handout Section ??.
 - a. Models factual relationships.
 - b. Possible to also model interrogative or modal with “fuzzy arrows.”
 - 2. Any appropriate category of worldviews will do.
 - B. What is a functor from this to **Set**?
 - IV. A simplicial complex of interaction
 - A. n -lects
 - 1. Each person has his or her own ideolect.
 - 2. Each interacting pair has a 2-lect.
 - 3. The difference between a 3-lect and three 2-lects.
 - 4. A snapshot or aggregate of human interaction as simplicial complex
 - B. Communication protocol. See handout Section ??.
 - 1. Common ground
 - a. Say two people: $K_A \leftarrow K_{AB} \rightarrow K_B$
 - b. A sends an idea to B .
 - 2. Give example
 - 3. Interpret / Learn / Reject
 - C. Sheaf of languages on simplicial complex
 - 1. Recall sheaf idea
 - 2. What is this whole object?
 - 3. How does information flow? Barwise?
- V. Future directions
 - A. My questions
 - 1. How does the network change through communication?
 - a. Each node/worldview can change
 - b. How can learning affect the shape of the network?
 - 2. Relation to brain: how does the network learn?
 - a. Hebb’s rule: Fire together – wire together.
 - b. In brain and in life: common experience creates connection.
 - 3. What does the network as a whole know?
 - 4. What is dishonesty and what does it do to the network’s ability to learn or understand?
 - B. Meta:
 - 1. I am interested in communicating across disciplines.
 - 2. I want to form a strong common ground with linguists.
 - 3. How to foster that?

1. SIMPLICIAL COMPLEXES

A simplicial complex is a polygonal shape of arbitrary (finite) dimension.

It consists of “convex hulls of vertices.” That is vertices, edges, filled-in triangles, filled-in tetrahedra, etc. A triangle that is not filled-in is also possible, but it’s simply three edges glued together.



The above depiction is ambiguous (because of the difficulties of drawing 3-dimensions on paper). Lets say that the $klmn$ tetrahedral shape in \mathcal{S} is not “filled in,” nor is its kmn triangle. Then there are

- (1) 14 vertices
- (2) 29 edges
- (3) 12 filled-in triangles
- (4) 2 filled-in tetrahedra.

Let’s make the definition rigorous.

Recall that every set X has a “power set,” denoted $\mathbf{P}(X)$, which is the set of subsets of X .

$$\mathbf{P}(\{1, 2, 3\}) = \{\{\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

Definition 1. A *simplicial complex* $\mathcal{S} = (S_0, S)$ consists of the following data:

- (1) A set S_0 whose elements are called *the vertices of \mathcal{S}* and
- (2) a collection of finite subsets $S \subset \mathbf{P}(S_0)$ of these vertices.
If $x \in S$ consists of n vertices of S_0 we call it an $n - 1$ -simplex.

These data must satisfy the following requirements:

- (1) If $x \in S$ is a simplex then every subset of x must be a simplex.
- (2) Every singleton (atom) is a simplex.

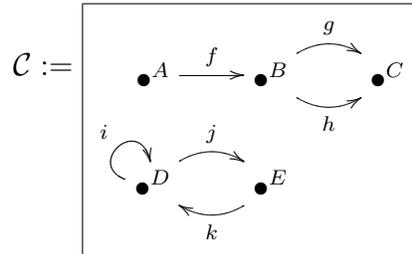
Example: $\mathcal{S} = (S_0, S)$ where $S_0 = \{f, g, h, i\}$ are the vertices and

$$S = \{\{g, h, i\}, \{g, h\}, \{g, i\}, \{h, i\}, \{f, g\}, \{f\}, \{g\}, \{h\}, \{i\}\}.$$

2. CATEGORIES

2.1. Preliminary idea and examples; formal definition.

- Idea: A category models objects of a certain sort and the relationships between them.
- These relationships are (or become) transitive.
- Examples:
 - People and heredity;
 - People and friendships??;
 - propositions and deductions;
 - places and routes.



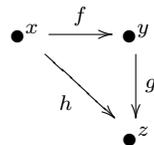
- Think of it like a graph: the nodes are objects and the arrows are relationships.
- Some paths can be equated with others (example: $j.k = i^3$).

Definition 2. A category \mathcal{C} consists of the following data:

- (1) A set $\mathbf{Ob}(\mathcal{C})$, called *the set of objects of \mathcal{C}* .
Objects $x \in \mathbf{Ob}(\mathcal{C})$ may be written as \bullet^x or simply as x .
- (2) For each $x, y \in \mathbf{Ob}(\mathcal{C})$ a set $\mathbf{Arr}_{\mathcal{C}}(x, y)$, called *the set of arrows in \mathcal{C} from x to y* .
An element of $\mathbf{Arr}_{\mathcal{C}}(x, y)$ may be written $f: x \rightarrow y$ or $\bullet^x \xrightarrow{f} \bullet^y$.
- (3) For each $x, y, z \in \mathbf{Ob}(\mathcal{C})$ a *composition law*

$$\mathbf{Arr}_{\mathcal{C}}(x, y) \times \mathbf{Arr}_{\mathcal{C}}(y, z) \rightarrow \mathbf{Arr}_{\mathcal{C}}(x, z).$$

We write $f.g = h$ to denote that following arrow f then following arrow g is the same as following arrow h . In this case we say the triangle



commutes.

These data must satisfy the following requirements:

- (1) Every object y has an “identity arrow” $\text{id}_y: y \rightarrow y$ such that $f.\text{id}_y = f$ and $\text{id}_y.g = g$ for all $f: x \rightarrow y$ and all $g: y \rightarrow z$.
- (2) Composition is *associative*. That is, given

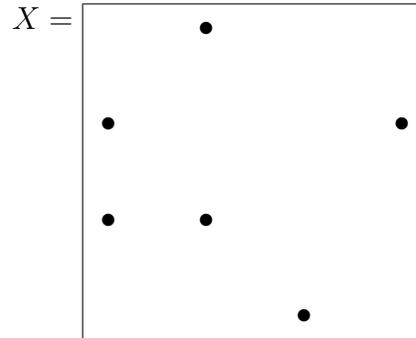
$$\bullet^w \xrightarrow{f} \bullet^x \xrightarrow{g} \bullet^y \xrightarrow{h} \bullet^z,$$

one has the equation

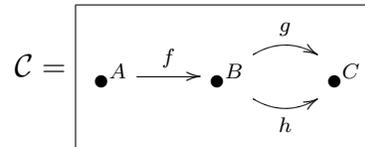
$$f.(g.h) = (f.g).h$$

2.2. Examples of categories.

- Any set X can be considered as a category. Its objects are the elements of X and it has no arrows (except an identity arrow for each object).



- Given a graph, make a category by adding in each path as a new “composite” arrow.



Here we have

- $\mathbf{Ob}(\mathcal{C}) = \{A, B, C\}$
- $\mathbf{Arr}_{\mathcal{C}}(A, B) = \{f\}$,
- $\mathbf{Arr}_{\mathcal{C}}(B, C) = \{g, h\}$,
- $\mathbf{Arr}_{\mathcal{C}}(A, C) = \{f.g, f.h\}$.

One could also declare $f.g = f.h$ and thereby get a different category.

- A partially ordered set (e.g. propositions and entailments) is a category: the elements of the set are objects and you put a single arrow $a \rightarrow b$ if $a \leq b$. Then transitivity becomes the composition law.
- The category **Set** of all sets. An object in **Set** is a set. An arrow $S \rightarrow T$ is a function assigning to each element of S an element of T .
- The category of geometric shapes. The objects are any geometric shape (in any precisely defined sense) and an arrow $S \rightarrow T$ is a continuous mapping.

3. FUNCTORS

3.1. Basic idea; formal definition.

- Idea: a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is a mapping between categories that preserves the stated structure.
- A functor takes objects to objects, arrows to arrows, and preserves the identities and the compositions.

Definition 3. Let \mathcal{C} and \mathcal{D} be categories. A *functor from \mathcal{C} to \mathcal{D}* , denoted $F: \mathcal{C} \rightarrow \mathcal{D}$, consists of the following data:

- (1) A function $F_0: \mathbf{Ob}(\mathcal{C}) \rightarrow \mathbf{Ob}(\mathcal{D})$ assigning to every object in \mathcal{C} some object in \mathcal{D} .
- (2) For every $x, y \in \mathbf{Ob}(\mathcal{C})$ a function $F_1: \mathbf{Arr}_{\mathcal{C}}(x, y) \rightarrow \mathbf{Arr}_{\mathcal{D}}(F_0(x), F_0(y))$.

[We write F to denote both F_0 and F_1 by abuse of notation.] These data must satisfy the following requirements:

- (1) For every $x \in \mathbf{Ob}(\mathcal{C})$, we must have $F(\text{id}_x) = \text{id}_{F(x)}$. “ F preserves identities.”
- (2) For all composable arrows $x \xrightarrow{f} y \xrightarrow{g} z$ in \mathcal{C} we have $F(f.g) = F(f).F(g)$:

$$\begin{array}{ccccc} \bullet x & \xrightarrow{f} & \bullet y & \xrightarrow{g} & \bullet z \\ & \searrow & & \nearrow & \\ & & f.g & & \end{array}$$

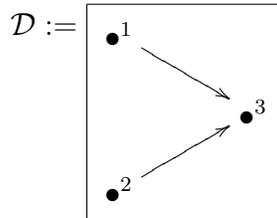
apply F , and ensure that composition is preserved.

$$\begin{array}{ccccc} \bullet F(x) & \xrightarrow{F(f)} & \bullet F(y) & \xrightarrow{F(g)} & \bullet F(z) \\ & \searrow & & \nearrow & \\ & & F(f).F(g)=F(f.g) & & \end{array}$$

“ F preserves composition.”

3.2. Examples.

- Consider the category with only one object (and its identity arrow), $\mathcal{C} = \boxed{\bullet}$. If \mathcal{D} is any category then a functor $\mathcal{C} \rightarrow \mathcal{D}$ is the same thing as an object in \mathcal{D} .
- Similarly, consider the category $\mathcal{C} = \boxed{\bullet \rightarrow \bullet}$. For any category \mathcal{D} , functor $\mathcal{C} \rightarrow \mathcal{D}$ is the same thing as an arrow in \mathcal{D} .
- Consider the set $\{a, b, c\}$ as a category \mathcal{C} . What is a functor $\mathcal{C} \rightarrow \mathbf{Set}$? Answer: three sets.
- Let \mathcal{D} be the category



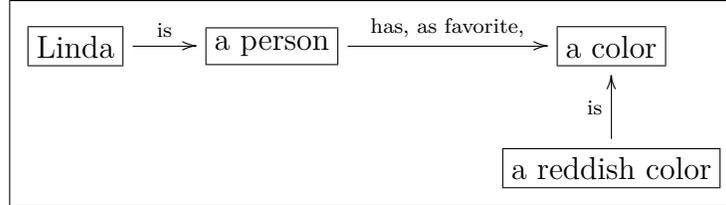
What is a functor $\mathcal{D} \rightarrow \mathbf{Set}$?

- Exercise: With \mathcal{C} and \mathcal{D} as above,
 - how many functors are there $\mathcal{C} \rightarrow \mathcal{C}$? Answer: 27.
 - how many functors are there $\mathcal{C} \rightarrow \mathcal{D}$? Answer: 27.
 - how many functors are there $\mathcal{D} \rightarrow \mathcal{C}$? Answer: 3.
 - how many functors are there $\mathcal{D} \rightarrow \mathcal{D}$? Answer: 9.

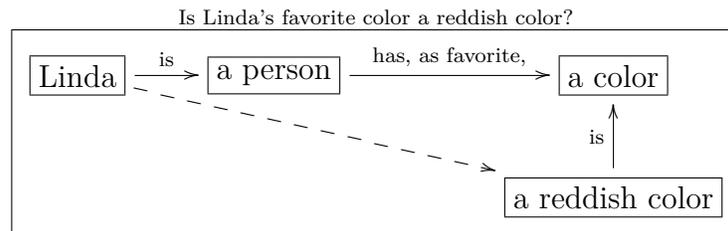
4. ONTOLOGIES (COGNITIVE CATEGORIES)

- A setting for discourse.

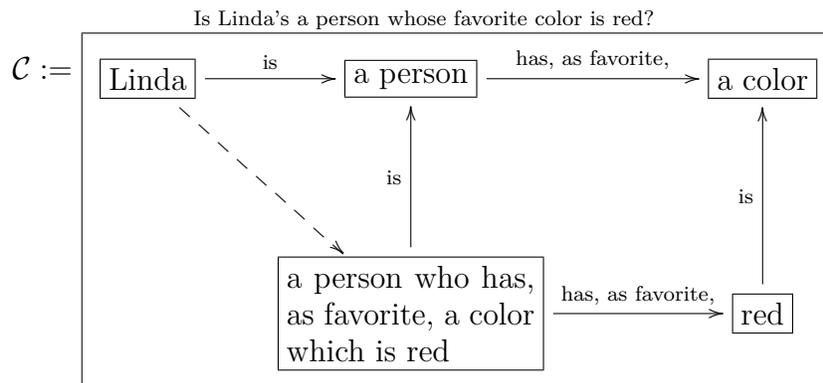
Linda is a person which has, as favorite, a color; a reddish color is a color.



- Interrogative:

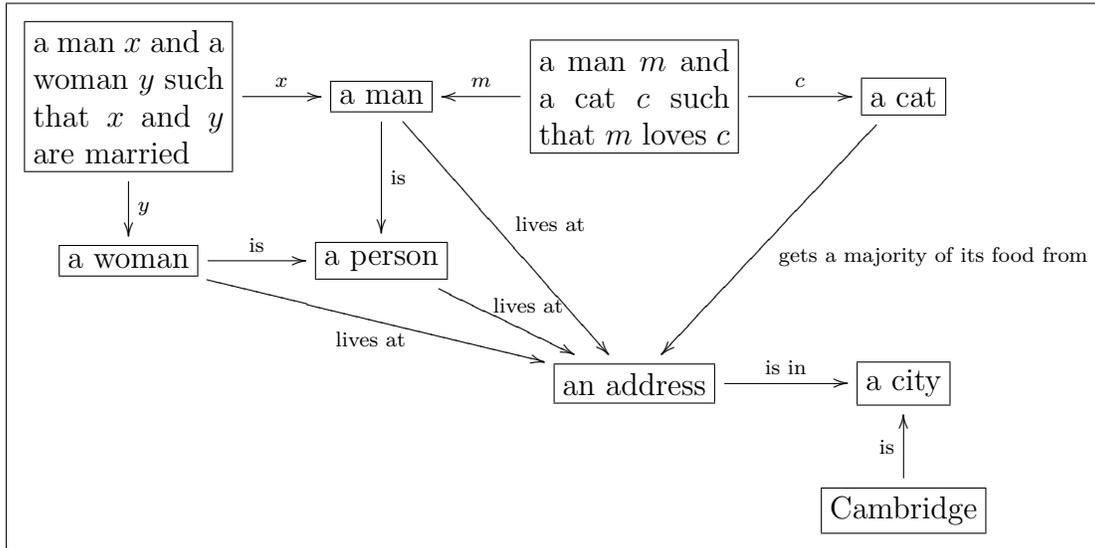


- Intersections (more precisely “fiber products”). In category theory, intersecting (or unioning) classes is common.



- By the way, what is a functor $\mathcal{C} \rightarrow \mathbf{Set}$?

- Associations or relations



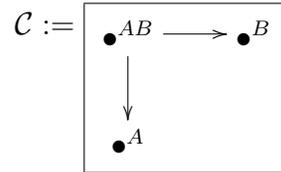
Q. Which of the diagrams above commute?

A. The upper left square does not commute for me! I don't know a priori whether, in your ontology, a man and a woman that are married must live at the same address. Similarly for a man and a cat he loves. So the above is not a category until those questions are resolved.

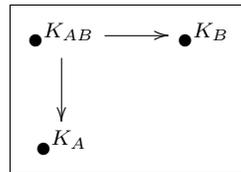
5. COMMUNICATION PROTOCOL

We've seen that categories can serve as "worldviews" or "knowledge bases." But maybe there is a more appropriate category of knowledge bases for some purpose. Let \mathcal{K} be any choice of one.

Consider the two-person network $\boxed{\bullet A \text{ --- } \bullet B}$. Its category of simplices is



A functor $F: \mathcal{C} \rightarrow \mathcal{K}$ determines a choice of three knowledge bases, one for A , one for B , and one for the "common ground," AB ; and F determines a choice of a mapping from the common ground to the knowledge base for A and another mapping to that of B . The image of F is



These knowledge bases can change while \mathcal{C} stays the same.

When Alice wants to communicate some new set of ideas (we'll call it \bar{I}) to Bob, she uses ideas (we'll call them I) from the common ground to support the new terminology and give it "a place to live." Bob then takes the new terminology and either interprets it within his framework, learns it as a new fact or definition, or rejects it as not fitting. (This could happen for example if Alice said that the man and woman in a married couple are the same person.) In fact, given a communication attempt, some aspects of it may be interpreted, some learned, and the rest rejected. Interpretation and learning can update the common ground as well as B 's knowledge base. The rejected portion does not change K_{AB} or K_B . In this way, A and B develop a rapport.

The situation above is modeled as the communication attempt

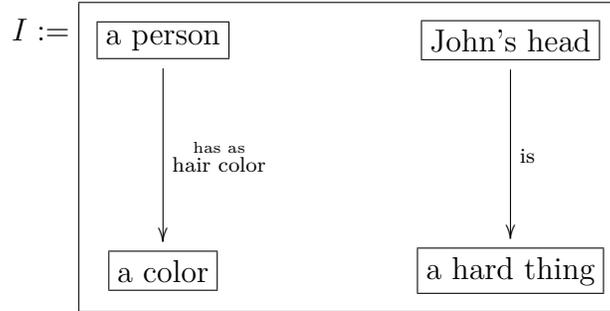
$$\begin{array}{ccccc} I & \longrightarrow & K_{AB} & \longrightarrow & K_B \\ \downarrow & & \downarrow & & \\ \bar{I} & \longrightarrow & K_A & & \end{array}$$

A picture of learning:

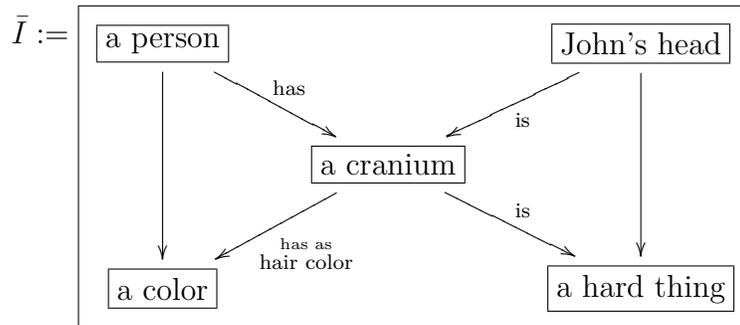
$$\begin{array}{ccccc} I & \longrightarrow & K_{AB} & \longrightarrow & K_B \\ \downarrow & & \downarrow & \lrcorner & \downarrow \\ \bar{I} & \longrightarrow & K'_{AB} & \longrightarrow & K'_B \\ \parallel & & \downarrow & & \\ \bar{I} & \longrightarrow & K_A & & \end{array}$$

Example 1. Suppose that A wants to communicate to B a fact about craniums, even though craniums are not part of the common knowledge between A and B . He wants to tell B : “Did you know that the color of a person’s hair can be determined by looking at their cranium? A cranium is a hard thing and John’s head is a cranium.”

Let I be the category



including in the obvious way to \bar{I} , which is the category



The above was for a single 2-way interaction, but in general we may have n -way interactions. This case is only slightly more involved for a simplicial complex.