Intuition/Talk on

String diagrams for traced and compact categories are oriented 1-cobordisms

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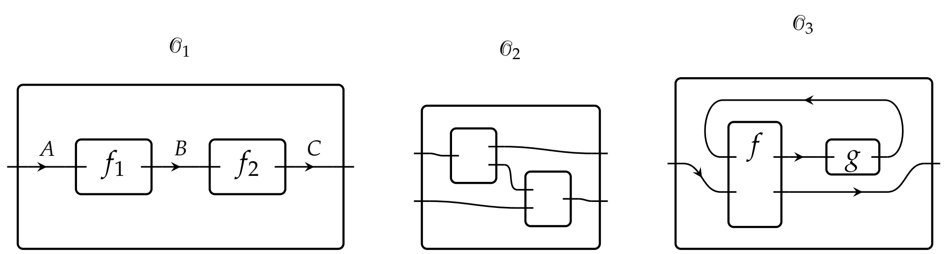
Abstract

We will present an alternate conception of string diagrams as labeled 1-dimensional oriented cobordisms, the operad of which we denote by $\mathbf{Cob}_{/\mathcal{L}}$, where $\mathcal{L}$ is the set of string labels. The axioms of traced (symmetric monoidal) categories are fully encoded by $\mathbf{Cob}_{/\mathcal{L}}$ in the sense that there is an equivalence between $\mathbf{Cob}_{/\mathcal{L}}$-algebras, for varying $\mathcal{L}$, and traced categories with varying object set. The same holds for compact (closed) categories, the difference being in terms of variance in $\mathcal{L}$. Time permitting, we will give a characterization of the 2-category of traced categories solely in terms of those of monoidal and compact categories, without any reference to the usual structures or axioms of traced categories.

Idea: operads can classify string diagram styles



for categories, monoidal categories, traced monoidal categories, etc. Then

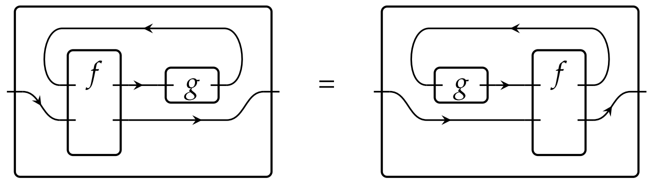
$$\mathrm{Lax}(\mathcal{O}, \mathbf{Set}) = \text{categories of sort } \mathcal{O}. \text{ "Doctrine"}$$

Here **Set** is acting as an enriching category.

Traced monoidal categories. Joyal, Street, Verity defined these. [All monoidal cats today are symmetric.] A *traced category* is a monoidal category $(\mathscr{C}, \otimes, I)$ + ... For all $X, Y, U \in \mathscr{C}$ a function

$$\mathrm{Tr}_{X,Y}^{U} \colon \mathrm{Hom}(X \otimes U, Y \otimes U) \to \mathrm{Hom}(X, Y)$$

satisfying certain axioms, e.g.

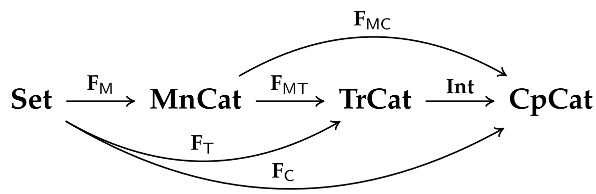


$$\mathrm{Tr}\left((g \otimes \mathrm{id}_Y) \circ f\right) = \mathrm{Tr}\left(f \circ (g \otimes \mathrm{id}_X)\right).$$

They also defined 2-adjunction

$$\mathbf{Int}\colon \boldsymbol{\mathfrak{TrCat}} \leftrightarrows \boldsymbol{\mathfrak{CpCat}} : U$$

This is part of a chain of adjunctions, with left adjoints shown below



A few more facts:

- The unit $T \to U\mathbf{Int}(T)$ is fully faithful for all traced T.
- Any monoidal M, compact C, and fully faithful $M \hookrightarrow C$ induces a unique trace structure on M for which the map is traced.
- The 2-categories $\boldsymbol{\mathfrak{MnCat}}$, $\boldsymbol{\mathfrak{TrCat}}$, and $\boldsymbol{\mathfrak{CpCat}}$ have (compatible) bo-ff orthogonal factorization systems.
- A map $F\colon T \to T'$ of traced categories is bijective-on-objects (resp. fully faithful) iff $\mathbf{Int}(F)\colon \mathbf{Int}(T) \to \mathbf{Int}(T')$ is.

Traceless characterization Let $\mathbf{F}_{\mathsf{M}}\colon \mathbf{Set} \to \boldsymbol{\mathfrak{MnCat}}$ and $\mathbf{F}_{\mathsf{C}}\colon \mathbf{Set} \to \boldsymbol{\mathfrak{CpCat}}$ be the free functors. The 2-cat $\boldsymbol{\mathfrak{TrCat}}$ of traced categories is bi-equivalent to $\boldsymbol{\mathfrak{TrFrObCat}}$, whose objects are

$$\begin{aligned}\mathrm{Ob}(\boldsymbol{\mathfrak{TrFrObCat}}) = \big\{(\mathcal{L}, C, F) \mid \mathcal{L} \in \mathbf{Set}, C \in \mathbf{CpCat}, F\colon \mathbf{F}_{\mathsf{M}}(\mathcal{L}) \to C,\\ \text{s.t. induced } F'\colon \mathbf{F}_{\mathsf{C}}(\mathcal{L}) \twoheadrightarrow C \text{ is bo}\big\}\end{aligned}$$

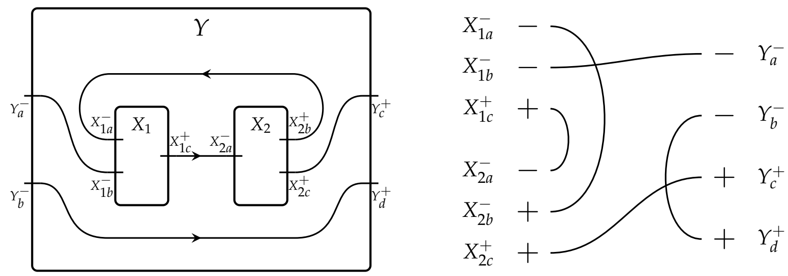
Morphisms in $\boldsymbol{\mathfrak{TrFrObCat}}$ are pairs $(g\colon \mathcal{L} \to \mathcal{L}', G\colon C \to C')$ making the diagram

$$\begin{array}{ccc} \mathbf{F}_{\mathsf{M}}(\mathcal{L}) & \xrightarrow{F} & C \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle G} \\ \mathbf{F}_{\mathsf{M}}(\mathcal{L}') & \xrightarrow[F']{} & C' \end{array}$$

commute. A 2-morphism $\alpha\colon (g_1, G_1) \to (g_2, G_2)$ in $\boldsymbol{\mathfrak{TrFrObCat}}$ exists only if $g_1 = g_2$, in which case it is a natural transformation $\alpha\colon G_1 \to G_2$ with no condition about diagrams commuting.

The map $\boldsymbol{\mathfrak{TrFrObCat}} \to \boldsymbol{\mathfrak{TrCat}}$ is given by factoring F as $\mathbf{F}_{\mathsf{M}}(\mathcal{L}) \xrightarrow{F_1} T \xrightarrow{F_2} C$, where F_1 is bijective-on-objects and F_2 is fully faithful. In this case, T is a traced category. The map $\boldsymbol{\mathfrak{TrCat}} \to \boldsymbol{\mathfrak{TrFrObCat}}$ is given by sending T to the composite $\mathbf{F}_{\mathsf{M}}(\mathrm{Ob}(T)) \to T \to \mathbf{Int}(T)$.

Cob is the operad for traced and compact categories.



Fully faithful $\mathbf{MnCat}^{\mathrm{lax}} \to \mathbf{Oprd}$. **Cob** as a monoidal category is sent to **Cob** as an operad. Since the functor is fully faithful, we can work either place. $\mathbf{MnCat}^{\mathrm{lax}}$ is more convenient, but **Oprd** has better pictures.

Some results Let $\mathcal{L} \in \mathbf{Set}$. We will show in below (0.0.2, 0.0.1) that there is an equivalence of categories

$$\mathrm{Lax}(\mathbf{Cob}/\mathcal{L}, \mathbf{Set}) \cong \mathbf{TrCat}_{\mathcal{L}}$$

where the latter is the 1-category of traced cats with object set free on $\mathcal{L}$.

More generally, let T be any traced category. Then we have an equivalence

$$\mathrm{Lax}(\mathbf{Int}(T), \mathbf{Set}) \cong \mathbf{TrCat}^{\mathrm{bo}}_{T/}$$

Similarly, let C be any compact category. We have an equivalence

$$\mathrm{Lax}(C, \mathbf{Set}) \cong \mathbf{CpCat}^{\mathrm{bo}}_{C/} \tag{1}$$

But putting these together we get something a little funny.

$$\mathbf{TrCat}^{\mathrm{bo}}_{T/} \cong \mathrm{Lax}(\mathbf{Int}(T), \mathbf{Set}) \cong \mathbf{CpCat}^{\mathrm{bo}}_{\mathbf{Int}(T)/}$$

Here's the idea. Given $T \twoheadrightarrow T'$ we get $\mathbf{Int}(T) \twoheadrightarrow \mathbf{Int}(T')$; that's one direction. For the other direction, given $\mathbf{Int}(T) \twoheadrightarrow C'$, we factor its composite with the unit $T \to \mathbf{Int}(T)$:

$$\begin{array}{ccc} T & \longrightarrow & \mathbf{Int}(T) \\ \downarrow\!\!\downarrow & & \downarrow \\ T' & \hookrightarrow & C' \end{array}$$

giving $T \twoheadrightarrow T'$.

Let's give intuition for (1).

Theorem 0.0.1.

$$\mathrm{Lax}(C, \mathbf{Set}) \cong \mathbf{CpCat}^{\mathrm{bo}}_{C/}$$

Proof intuition.
$\Leftarrow$: Suppose given $F\colon C \twoheadrightarrow C'$, bijective on objects. We construct $L_F\colon C \to \mathbf{Set}$ by

$$L_F(c) := \mathrm{Hom}_{C'}(I, c)$$

where c is an object or morphism of C' and I is the monoidal unit in C'. The lax structure is

$$L_F(c) \times L_F(c') = \mathrm{Hom}(I, c) \times \mathrm{Hom}(I, c') \xrightarrow{\otimes} \mathrm{Hom}(I \otimes I, c \otimes c') = L_F(c \otimes c').$$

$\Rightarrow$: Suppose given $L\colon C \to \mathbf{Set}$. We want $F_L\colon C \twoheadrightarrow C'$. Since its bijective on objects, we can put $\mathrm{Ob}(C) = \mathrm{Ob}(C')$, what about the morphisms of C'?

$$\mathrm{Hom}_{C'}(c_1, c_2) = L(c_1^* \otimes c_2).$$

Composition comes from the lax structure. $\square$

Corollary 0.0.2. *Since* $\mathbf{Cob}/\mathcal{L}$ *is the free compact category on* $\mathcal{L} \in \mathbf{Set}$*, it follows that*

$$\mathbf{TrCat}_{\mathcal{L}} \cong \mathrm{Lax}(\mathbf{Cob}/\mathcal{L}, \mathbf{Set}) \cong \mathbf{CpCat}_{\mathcal{L}}$$