BIFURCATION THEORY OF NETWORKED DYNAMICAL SYSTEMS

David I. Spivak

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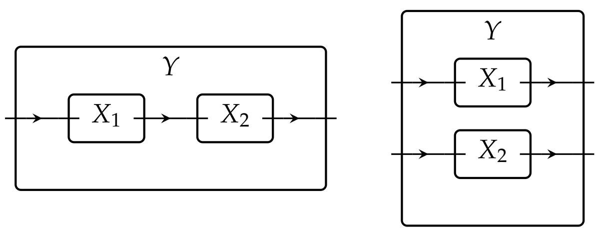
Summary

The steady states of an open dynamical system (one having input parameters and output values) are typically organized as a bifurcation diagram, which can be conceived of as a matrix of uncountable size. The reason for doing so is that the steady states of coupled dynamical systems can then be obtained by matrix arithmetic: multiplication for series composition, Kronecker (tensor) product for parallel composition, and partial trace for feedback. With this approach, the time required to calculate the steady states of a system of systems is reduced exponentially.

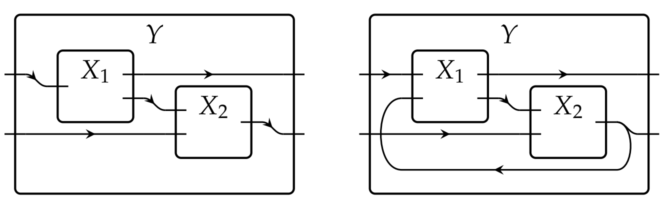
Introduction

Open dynamical systems can be composed to make larger systems. For example, they can be put together in series or in parallel



(1)

or in a more complex combination, possibly with feedback and splitting wires



(2)

A dynamical system has a set or space of states and a rule for how the state changes in time. An *open* dynamical system also has an interface X (as shown above), which indicates the number of input ports and output ports that exist for the system. Signals passed to the system through its input ports influence how the state changes. An output signal is generated as a function of the state and is passed through output port to serve as an input to a neighboring system. Here we can consider discrete or continuous models of dynamical systems, but for concreteness, we will consider the continuous model. If $A = \mathbb{R}^k$ is the space of input parameters and $B = \mathbb{R}^\ell$ is the set of output values, then a *dynamical system* of type $X = (A, B)$ consists of a state space $S = \mathbb{R}^n$ and a system of differential equations

$$\begin{aligned} \dot{s}_1 &= f_1(s_1, \ldots, s_n, a) \\ \dot{s}_2 &= f_2(s_1, \ldots, s_n, a) \\ \vdots\ & \ \vdots \qquad \vdots \\ \dot{s}_n &= f_n(s_1, \ldots, s_n, a) \end{aligned}$$

or simply $\dot{s} = f(s, a)$ as well as a readout map

$$b = g(s_1, \ldots, s_n)$$

or simply $b = g(s)$. We say that dynamical systems of this type *inhabit* a box $A \to \square \to B$.

For any interface X, let $\mathsf{OS}(X) = \{(S, f, g) \text{ as above}\}$ denote the set of all possible open dynamical systems of type X. The idea is that a diagram, such as any of those found in (1) or (2), determines a function

$$\mathsf{OS}(X_1) \times \mathsf{OS}(X_2) \to \mathsf{OS}(Y).$$

This function amounts to a formula that produces an open system of type Y given open systems of type X_1 and X_2, arranged—in terms of how signals are passed—according to the wiring diagram. The formula enforces that wires connecting interface correspond to variables shared by the dynamical systems

Compositional viewpoints of dynamical systems

We are interested in looking at open dynamical systems in ways that respect arbitrary interconnection (variable coupling) via wiring diagrams, as we now briefly explain. Above, we said that if open systems inhabit each interior box in a wiring diagram, we can construct a composite open system for the outer box. But this is true in other domains as well, such as matrices. That is, to each interface X, one can assign a set $\mathsf{Mat}(X)$ of the associated type; then, given a matrix in each interior box of a wiring diagram one can put them together to form a matrix for the outer box. In other words, a wiring diagram, such as any found in (1) or (2), should determine a function

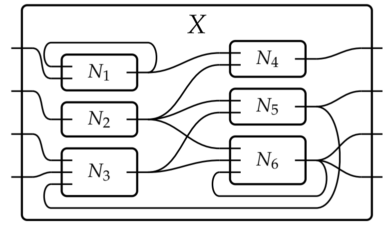
$$\mathsf{Mat}(X_1) \times \mathsf{Mat}(X_2) \to \mathsf{Mat}(Y).$$

Moreover, there is a compositional mapping $\mathsf{Stst}_X : \mathsf{OS}(X) \to \mathsf{Mat}(X)$, given by arranging steady states of a dynamical system into a matrix form. We say that a mapping is *compositional* if it behaves correctly with respect to wiring diagrams in the following sense. Given open systems of type X_1 and X_2, we can either compose first and apply the mapping to the result, or apply the mapping first and then compose the results. We want these to give the same answer. Formally, we express this by requiring the following diagram to commute:

$$\begin{array}{ccc} \mathsf{OS}(X_1) \times \mathsf{OS}(X_2) & \longrightarrow & \mathsf{OS}(Y) \\ {\scriptstyle \mathsf{Stst}_{X_1} \times \mathsf{Stst}_{X_2}} \downarrow & & \downarrow {\scriptstyle \mathsf{Stst}_Y} \\ \mathsf{Mat}(X_1) \times \mathsf{Mat}(X_2) & \longrightarrow & \mathsf{Mat}(Y) \end{array}$$

In this talk, we discuss a compositional mapping from open dynamical systems of several sorts—discrete, measurable, and continuous—to the matrix domain. The entries of these matrices list, count, or measure the steady states—also known as equilibria or fixed points—of the dynamical system for each input and output. The topology of a dynamical system is to a large degree determined by its set of steady states and their stability properties, and these are generally organized into *bifurcation diagrams* (e.g., as in [Str94]). The reason we refer to them as matrices, rather than as bifurcation diagrams, is that—as we show in this talk—bifurcation diagrams compose according to matrix arithmetic. That is, when several dynamical subsystems are put together in series, parallel, or with feedback to form a larger system, the classifying matrix for the whole can be computed by multiplying, tensoring, or computing a partial trace (adding up diagonal entries) of the subsystem matrices. There is also a matrix formula for computing the stability properties at each fixed point of the composed system using derivatives of the functions found in each subsystem.

This approach promises to reduce the complexity of computing the bifurcation theory for systems of systems. This is most easily explained for interconnected systems of discrete dynamical systems. The state transition table for such a system grows exponentially in both the number of input wires and the number of internal boxes. In contrast, the matrix of steady states grows only in the number of input wires.



(3)

For example, if each input wire in the diagram above carries two signals (say 'resting' or 'active'), and each box carries three states (e.g., 'depolarized', 'polarized', or 'hyperpolarized') then to express the totalized dynamical system would require a table with roughly $2^4 3^6 = 11,648$ rows, whereas the matrix of steady states would require a relatively small 16×16 matrix. As more internal boxes are encapsulated by the wiring diagram, an exponential savings is achieved by considering the steady state matrix, rather than the whole dynamical system.

A potential interpretation of the steady state matrix in neuroscience is as follows. In perception, it is not uncommon to consider neurons as dynamical systems [Izh07], and input signals can be classified as either expected or unexpected [CF78]. One way to think about this is that expected input signals are those that do not change the state of the system, or at least do not change it by very much. When the state is unchanged, so is the output of the system i.e., expected perception does not cause a change in behavior. The steady state matrix presented here measures, for each (perception, behavior) pair, the set of states that are expected in that context. The purpose of the present paper is to show that this measurement is compositional, i.e., that it respects any given wiring structure.

References

[CF78] Edward C. Carterette and Morton P. Friedman, eds. *Handbook of Perception*. Vol. IX. Perceptual Processing. Academic Press, 1978, p. 433.

[Izh07] Eugene M. Izhikevich. *Dynamical Systems in Neuroscience*. MIT Press, 2007, p. 441.

[Str94] Steven H. Strogatz. *Nonlinear Dynamics and Chaos*. Studies in nonlinearity. Perseus Books, 1994, p. 498. ISBN: 0-201-54344-3.