Calculating steady states of nonlinear dynamical systems using matrix arithmetic

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Abstract

Open dynamical systems are mathematical models of machines that take input, change their internal state, and produce output. For example, one may model anything from neurons to robots in this way. Several open dynamical systems can be arranged in series, in parallel, and with feedback to form a new dynamical system—this is called compositionality—and the process can be repeated in a fractal-like manner to form more complex systems of systems.

I will discuss a technique for calculating the steady states of an interconnected system of systems, in terms of the steady states of the component dynamical systems. The steady states, or equilibria, are organized into "steady state matrices" which generalize bifurcation diagrams. I'll show that the compositionality structure of dynamical systems fits with familiar operations on matrices: serial, parallel, and feedback compositions correspond to multiplication, Kronecker product, and partial trace operations on matrices. Thus we can calculate the steady states of a system of dynamical systems by doing matrix arithmetic on the individual steady state matrices. This talk will be aimed at an undergraduate level.

Outline

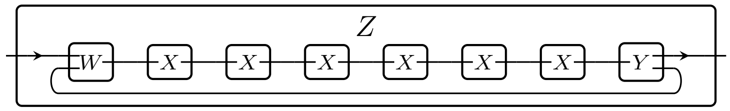
Thanks to Dikran Karagueuzian for the invitation.

I. Introduction
   A. High-level view of dynamical systems
   B. Ubiquity of coupled dynamical systems
   C. Compositional properties
   D. Applied category theory under the hood
   E. Outline: Dynamical systems, wiring diagrams, compositional mappings

II. Dynamical systems
   A. Continuous
      1. Autonomous vs. open
      2. Steady states = bifurcation theory
   B. Discrete (open; steady states)

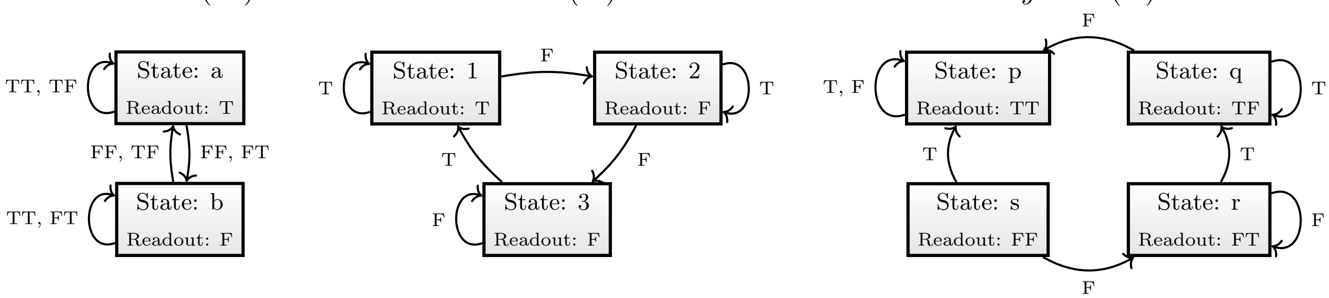
III. Wiring diagrams and black-boxing
   A. Wiring diagrams: intuitive
      1. Boxes, wiring diagrams, and nesting
      2. Inhabitants and black-boxing formulas
      3. Special cases: serial, parallel, feedback
      4. Challenge: see Figure 1.
   B. A categorical aside
      1. Dynamical systems *do not* quite form a (traced) category.
      2. They form an algebra $\mathrm{DS}\colon \mathcal{W} \to \mathbf{Set}$ on an operad $\mathcal{W}$.
      3. Compositional mappings are morphisms of algebras.

IV. Compositional mappings
   A. Idea: Averages vs. weighted averages
   B. Euler's method
   C. Steady state matrices: solve challenge
   D. Continuous case: pixelate



(1)

$w \in \mathrm{DS}(W)$ $\quad$ $x \in \mathrm{DS}(X)$ $\quad$ $y \in \mathrm{DS}(Y)$



The composed dynamical system $z \coloneqq \mathrm{DS}(\varphi)(w, x, x, x, x, x, x, y)$ has $2 \cdot 3^6 \cdot 4 = 5832$ states.

$\mathrm{StstS}(w) =$

Outputs: / Is fixed by:	T	F
TT	$\{a\}$	$\{b\}$
TF	$\{a\}$	$\emptyset$
FT	$\emptyset$	$\{b\}$
FF	$\emptyset$	$\emptyset$

$\mathrm{StstS}(x) =$

Outputs: / Is fixed by:	T	F
T	$\{1\}$	$\{2\}$
F	$\emptyset$	$\{3\}$

$\mathrm{StstS}(y) =$

Outputs: / Is fixed by:	TT	TF	FT	FF
T	$\{p\}$	$\{q\}$	$\emptyset$	$\emptyset$
F	$\{p\}$	$\emptyset$	$\{r\}$	$\emptyset$

$\mathrm{Tr}\left(\mathrm{StstS}(w)\mathrm{StstS}(x)^6\mathrm{StstS}(y)\right) =$

Outputs: / Is fixed by:	T	F
T	$\left\{\begin{array}{l} a111111p, a111112p, \\ a111123p, a111233p, \\ a112333p, a123333p, \\ a233333p, b333333p, \\ a111111q \end{array}\right\}$	$\left\{\begin{array}{l} a111112r, a111123r, \\ a111233r, a112333r, \\ a123333r, a233333r, \\ b333333r \end{array}\right\}$
F	$\{b333333p\}$	$\{b333333r\}$

Figure 1: Example of steady state matrix calculation