A whirlwind tour of category theory, for the working scientist and engineer

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Outline

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1. **Introduction**
   - What is category theory?
   - Analogies
   - Plan of the talk
2. Categorification
3. Something more concrete
4. Defining categories and functors
5. The whirlwind tour

What is category theory?

- Since its invention in the early 1940s, category theory (CT) has revolutionized math.
- It's a principled way of thinking about mathematical subjects.
- CT was invented to build bridges between disparate branches of math.
  - It does so by distilling the essence of mathematical structure.
  - We consider: the objects of study and the relationships between them.

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  - It does so by distilling the essence of mathematical structure.
  - We consider: the objects of study and the relationships between them.
- Original use: connect two disparate subjects, topology and algebra.
  - The essence of each subject was formulated as a category.
  - Rigorous relationships (functors) were established between them.
  - Functors allowed the import of theorems from algebra into topology, as *new theorems*.

Analogies

- Category theory interconnects mathematics.
  - Whole disciplines, subdisciplines, or simple thoughts can be categories.
  - Functors connect categories, building bridges between worlds.
- Distinct towns and bridges; not sprawl.

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  - We don't need everything to be put into one big pile.
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  - Often two fields share some common substructures.
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  - Functors make this notion of commonality precise.
  - Connectable disciplines may be neighboring, distant, or fractal.
- Connecting disciplines is very important for solving hard problems.
- In that spirit, let's make some analogies.

Relativity and quantum mechanics

- Category theory has something in common with relativity.
  - Set theory is akin to Aristotilian space-time: one reference frame.
  - A category is a reference frame in which one sees some part of math.
  - Category theory is a multiplicity of interconnected perspectives.

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- Category theory has something in common with quantum mechanics (QM).
  - What might a non-physicist secretly think about QM?
  - It's strange, esoteric, possibly wrong or incomplete, not relevant to me.
  - But note that QM is ubiquitous to the point of mundane:
    - Lasers at the grocery store.
    - LEDs in your flashlight.
    - Transistors in your phone.
    - You can't create any of these without a deep understanding of QM.

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- In modern mathematics, CT is ubiquitous and essential.
  - It might seem strange at first, but it's not going away.
  - And scientists and engineers are starting to realize its value.

Category theory: branching out

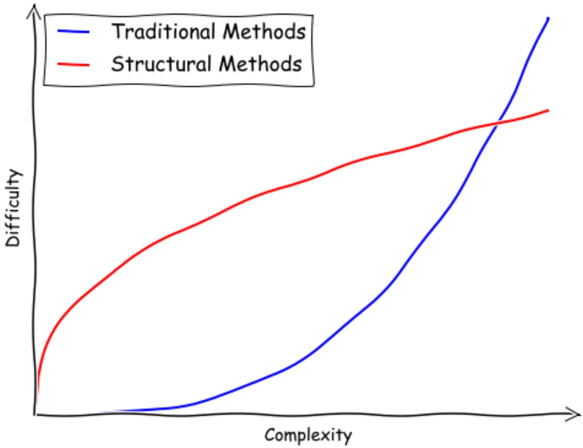
- Category theory naturally fosters connections between disparate fields.
- It has branched out of math and into:
  - physics,
  - linguistics,
  - materials science,
  - biology,
  - manufacturing,
  - economics and game theory, etc.
- CT has had most success in computer science, formalizing programming languages.

Category theory at NIST

NIST is the US national institute of standards and technology.

- They are concerned with
  - Information representation
  - Model integration
  - Multiple scales and formalisms
- They have identified category theory as a potential foundation.
- Here's how they put it, in cartoon form:

NIST: "The need for categorical engineering"



Breiner, S., Jones A., Subrahmanian, E. "Categories for the working engineer: A call for applied category theory." *International Category Theory Conference 2016*, Halifax.

Applications we'll discuss today

In the abstract to the talk I mentioned:

- hierarchies,
- symmetries,
- agent actions,
- information and communication,
- local-to-global relationships, and
- compositionality.

We'll get to each of these concepts along the way, in no particular order.

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Categorification in mathematics

- Over the last 20 years, many fields of math have been *categorified*.
  - This means reformulated at a “higher categorical level”.
  - Equality is replaced by isomorphism.
  - This program has led to many new results across mathematics.
- Is $2+2$ equal to 4 or are they just isomorphic?
  - Let’s look at our hands.
  - Can the difference possibly matter?

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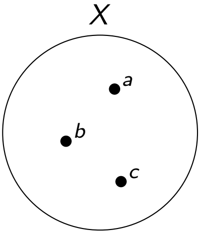
- Is 2+2 equal to 4 or are they just isomorphic?
  - Let's look at our hands.
  - Can the difference possibly matter? The answer is yes!
- We will begin by categorifying the natural numbers:

$$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots, 871^{23}, \ldots, n, \ldots\}$$

Finite sets

I assume everyone knows what a set is, but let's recall.

- A set is a bag of dots.
  - The set $X = \{a, b, c\}$ is shown to the right.
  - An element of X is denoted $x \in X$.
  - The *cardinality* of X is the number of elements, denoted $|X|$. Here $|X| = 3$.
  - The *empty set* $\emptyset$ is a bag with no dots.
- Finiteness: X is finite; $\mathbb{N} = \{0, 1, 2, \ldots\}$ is not.
- For any $n \in \mathbb{N}$, define $\underline{n} := \{1, \ldots, n\}$, finite!
  - So, for example, $\underline{0} = \emptyset$.



Categorification idea: what we call "numbers" are just cardinalities.

Functions

- A function from X to Y is an assignment:
  - Each $x \in X$ is assigned exactly one $y \in Y$.
  - We write $x \mapsto y$ and say x is *sent to* y.
- If the function is called f, we write $f\colon X \to Y$ or $X \xrightarrow{f} Y$,
  - and $f(x) = y$ means f sends x to y, or $f\colon x \mapsto y$.

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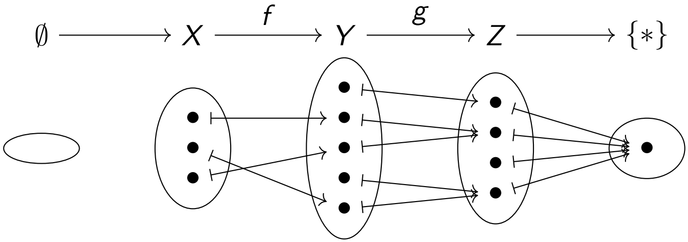
- Functions can be composed $X \xrightarrow{f} Y \xrightarrow{g} Z$, denoted $X \xrightarrow{f;g} Z$.



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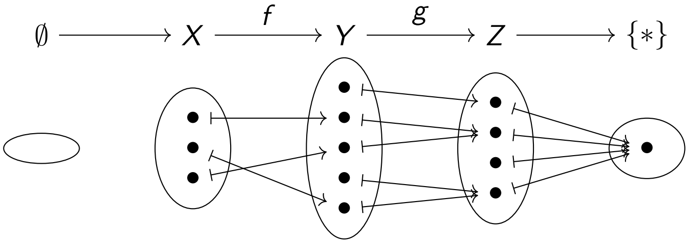

Identity and isomorphism

- For any set X, there is a function $\mathrm{id}_X \colon X \to X$.
    - It is defined by $\mathrm{id}_X(x) = x$ for any $x \in X$.
    - It's a unit for composition: $f ; \mathrm{id} = f$ and $\mathrm{id} ; f = f$.
- An *isomorphism* from X to Y is
    - a function $f \colon X \to Y$ and a function $g \colon Y \to X$
    - such that $f ; g = \mathrm{id}_X$ and $g ; f = \mathrm{id}_Y$.
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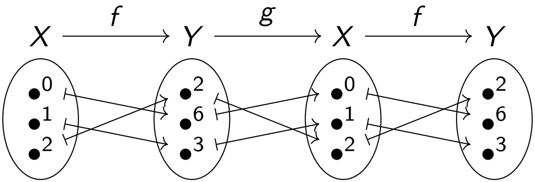

Cantor's definition of cardinality: isomorphism

- So what is the cardinality $|X|$ of a finite set X?
  - Its the $n \in \mathbb{N}$ such that $X \cong \{1, 2, \ldots, n\}$.
  - We recover cardinality from the notion of isomorphism.
- For arbitrary A and B, how many isomorphisms $A \to B$ are there?
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  - If A and B have different number of elements? None!
  - If they both have n elements? $n!$.
- Several more slides about relatively straightforward math.
  - Goal: show that sets and functions are primary.
  - Numerical operations are secondary, following from a deeper story.

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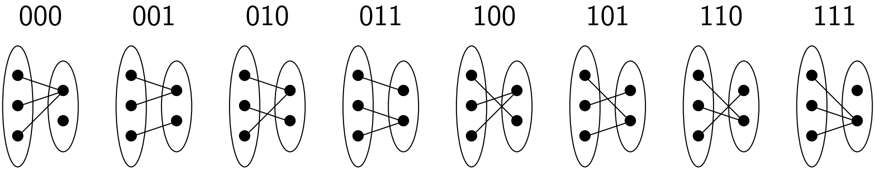

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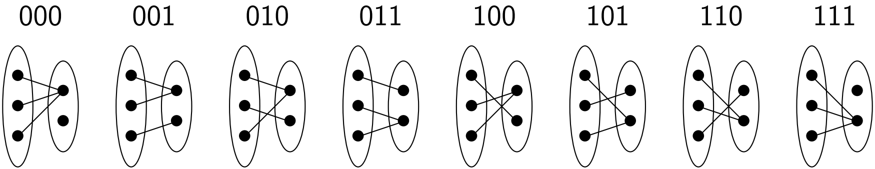

- In general there are n^m functions $\underline{m} \to \underline{n}$.
- Does it always work?
  - Are there really 1^m functions $\underline{m} \to \underline{1}$?
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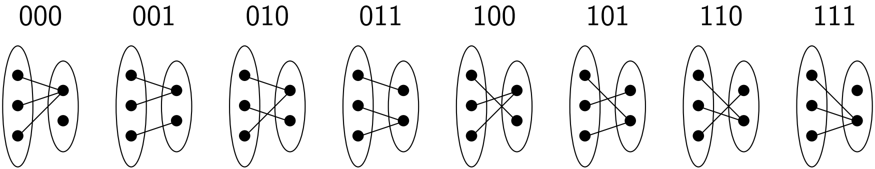

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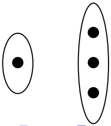
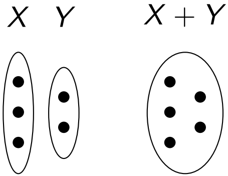
$\underline{1} \longrightarrow \underline{n}$



Disjoint unions

The disjoint union of sets X and Y is given by lumping them together.

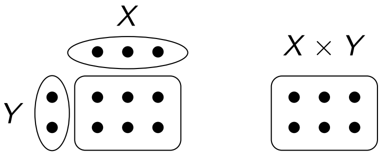


- Disjoint union, also called *coproduct*, is an operation on pairs of sets.
  - For any sets X and Y, we have a new set $X + Y$.
  - It categorifies the $+$ operation on natural numbers.
  - It is associative and commutative, and $\underline{0}$ is a unit.

$$X+Y \cong Y+X \qquad X+(Y+Z) \cong (X+Y)+Z \qquad X+\underline{0} \cong X \cong \underline{0}+X$$

Products

The *product* of sets X and Y is given by making a grid.



- Product is an operation on pairs of sets.
  - For any sets X and Y, we have a new set $X \times Y$.
  - It categorifies the $\times$ operation on natural numbers.
  - It is associative and commutative, and $\underline{1}$ is a unit.
  - It "distributes over addition" in the usual way.

$$X \times Y \cong Y \times X \qquad X \times (Y \times Z) \cong (X \times Y) \times Z \qquad X \times \underline{1} \cong X \cong \underline{1} \times X$$

$$X \times (Y + Z) \cong (X \times Y) + (X \times Z) \qquad X \times \underline{0} \cong \underline{0} \cong \underline{0} \times X$$

What more does arithmetic tell us?

We now have categorified 0, 1, $+$, $\times$, and ˆ.

- Category theory says that mappings govern all. So ˆ is paramount.
  - The fact that $1^n \cong 1$ means there is exactly one map $\underline{n} \to \underline{1}$.
  - The fact that $n^0 \cong 1$ means there is exactly one map $\emptyset \to \underline{n}$.
  - These *define* 0 and 1 in any category!
- So what do the following arithmetic facts tell us?

$$m^a \times m^b = m^{a+b} \qquad\qquad m^a \times n^a = (m \times n)^a$$

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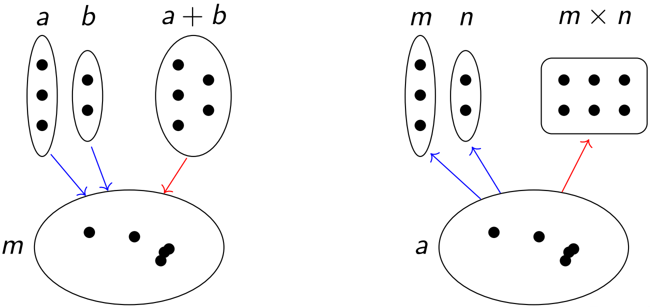

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- Is this just a slogan? Is this buddhist postmodernist hippy nonsense?
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  - No, it's called "abstract nonsense": concrete essences abstracted away.
  - But it's not nonsense: it's caused a paradigm shift within math.
  - And you can see for yourself: let's recap and then move on.

Recapping

The take-away so far is that:

- Numbers $0, 1, 2, \ldots$ are shadows on Plato's cave.
  - Behind them lurk finite sets $\{\}, \{1\}, \{a, b\}$.
  - The finite *set* of people in this room is far richer than the *number*.
- The constants $0, 1$ and operations $+, \times, \hat{}$ are also shadows.
  - They are completely defined by relationships, e.g. $m^{a+b} = m^a \times m^b$.
  - Every set is completely determined by the maps into and out of it.

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So what is categorification?

It's searching for the relationships that explain the shadows in Plato's cave.

Some other categories

A category is roughly a bunch of objects and relationships.

- We just discussed the category with finite sets and functions.
- There is also the category of vector spaces and linear transformations.
  - What are the "cardinalities" in this case? $\mathbb{R}^n$ takes the place of $\underline{n}$.
  - The isomorphisms are the invertible matrices.
  - One can take products and sums of vector spaces.
- There is the category of graphs and graph homomorphisms.
- There is the category of databases and database mappings.
- There is the category of data types and programs.

In each case the above story can be found hidden inside.

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- And then there are custom categories.

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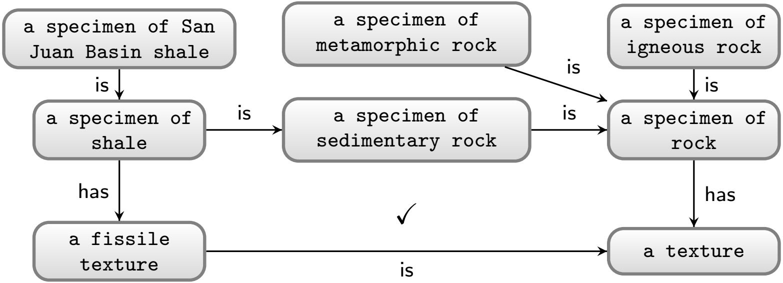

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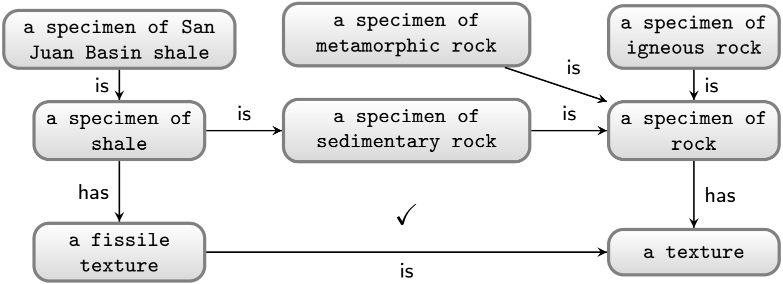

Concept webs can be useful. But are they math?

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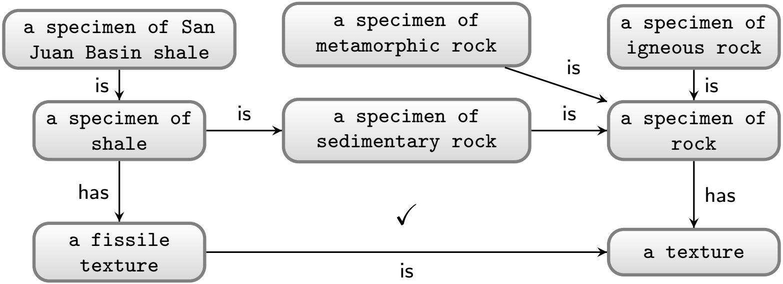

- One way to make this mathematical is to require that:
  - every box can be assigned a finite set of examples.
  - every arrow can be assigned a function between the finite sets.
  - the two functions ⌜shale⌝ $\to$ ⌜texture⌝ should agree (hence the ✓).

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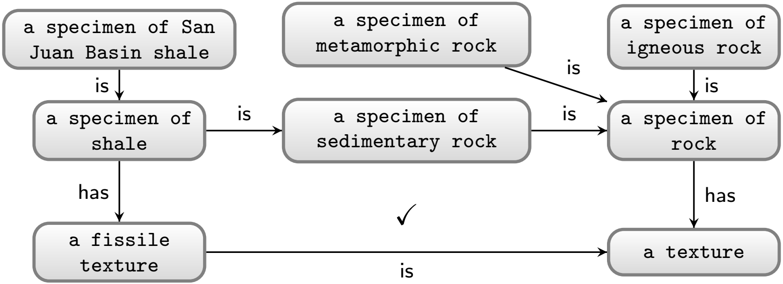

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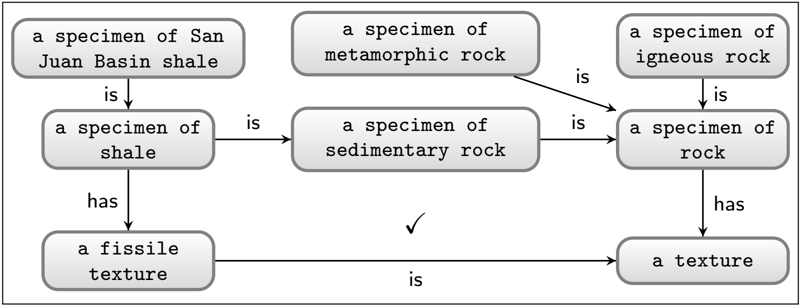
  - the two functions $\ulcorner\text{shale}\urcorner \to \ulcorner\text{texture}\urcorner$ should agree (hence the ✓).
- Perhaps also that there is a coproduct:

$$\ulcorner\text{igneous}\urcorner + \ulcorner\text{metamorphic}\urcorner + \ulcorner\text{sedimentary}\urcorner \cong \ulcorner\text{rock}\urcorner.$$

A functor connects our schema to data

We have built a custom category:

$$\mathcal{C} =$$



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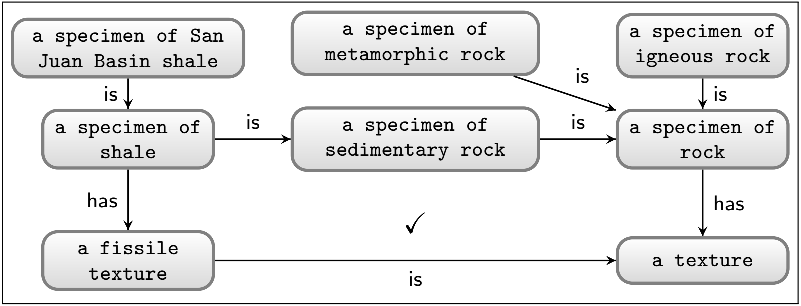

and we give it meaning by associating a functor $\mathcal{C} \to$ **FinSet**.

Let's back up a little

- I want to define categories and functors so that this makes sense.
- Then we'll have our whirlwind tour of examples.

Outline

A category is a graph with extra structure

Recall that a graph (a.k.a. directed multigraph) consists of

- A bunch of dots $\bullet$, and a bunch of arrows $\bullet^a \to \bullet^b$.
- There can be several arrows $\bullet^a \rightrightarrows \bullet^b$, and even arrows $\bullet^a \to \bullet^a$.
- A path is a sequence $\bullet^{a_0} \to \bullet^{a_1} \to \cdots \to \bullet^{a_n}$.
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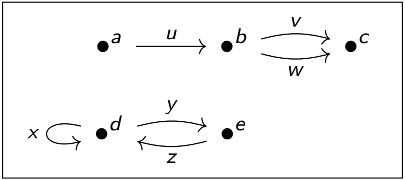

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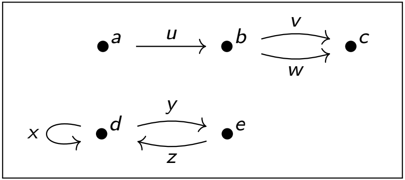
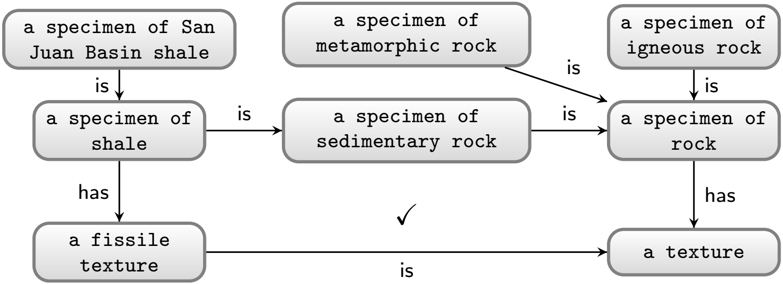


A category is a graph *plus* the ability to declare two paths equivalent.

- So we could declare $u; v \simeq u; w$ and/or $x; x; x \simeq \mathrm{id}_d$.
- The concatenations of equivalent paths are equivalent.

Back to the shale category



- This graph happens to have eight vertices and eight arrows.
  - Here we know that every rock has a texture,
  - some textures are fissile, shale is a rock, etc.

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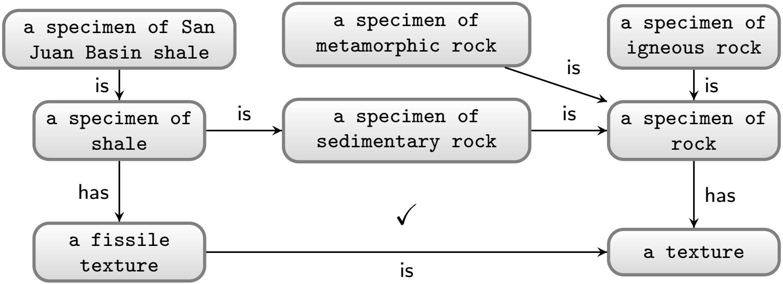

- This graph happens to have eight vertices and eight arrows.
  - Here we know that every rock has a texture,
  - some textures are fissile, shale is a rock, etc.
- To make it a category we can add any path equivalences we want.
  - We want to say that shale's texture is fissile.
  - It follows that the texture of San Juan shale is also fissile.

The category of finite sets

We said that there is a category **FinSet** of finite sets.

- There is a vertex (usually called an *object*) for each finite set A.

$$\bullet^{\underline{17}} \qquad \bullet^{\{a,b,c\}}$$

- There is an arrow (also called a *morphism*) for each function $f\colon A \to B$.
- Two paths are equivalent if their composite is the same function.

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Just to be clear, **FinSet** is a huge category.

- There are 871^{23} arrows from vertex $\underline{23}$ to vertex $\underline{871}$.
- There are many paths from $\underline{23}$ to $\underline{1}$, but they are all equivalent.

A functor is a sort of graph mapping

- Suppose we have two categories $\mathcal{C}$ and $\mathcal{D}$.
- We want to connect them by a functor $F\colon \mathcal{C} \to \mathcal{D}$.
- A functor is like a function, except it has to deal with more structure.
  - F sends the vertices in $\mathcal{C}$ to the vertices in $\mathcal{D}$.
  - F sends the arrows in $\mathcal{C}$ to *paths* in $\mathcal{D}$.
  - So a path p in $\mathcal{C}$ is also sent to a path $F(p)$ in $\mathcal{D}$.
  - And if $p \simeq q$ are equivalent paths in $\mathcal{C}$, then $F(p) \simeq F(q)$ in $\mathcal{D}$.

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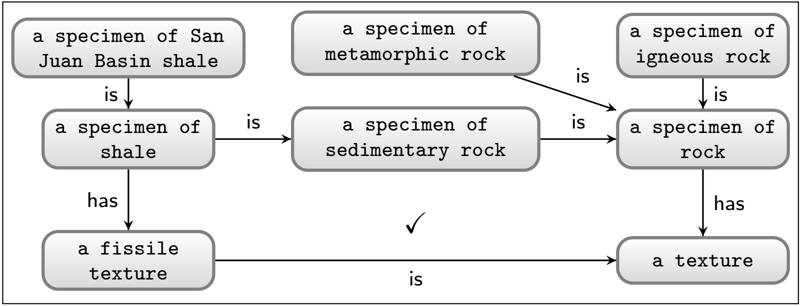
- So a functor $\mathcal{C} \to$ **FinSet** consists of:
  - A finite set $F(c)$ for every vertex $c \in \mathcal{C}$,
  - a function $F(c) \to F(d)$ for every arrow $c \to d$ in $\mathcal{D}$,
  - and if two paths are equivalent $p \simeq q$ in $\mathcal{C}$, then $F(p)$ and $F(q)$ must be the same function.

Shale data

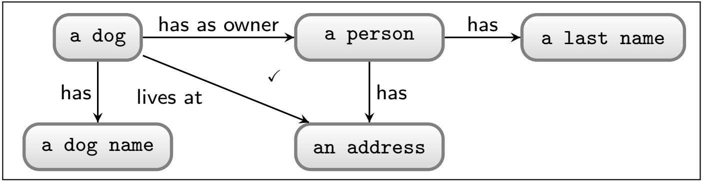
$\mathcal{S} =$



What is a functor $\mathcal{S} \to$ **FinSet**?

- Imagine the set of all shale specimens and that of all rock specimens.
- Imagine is a function $\ulcorner$shale$\urcorner \to \ulcorner$rock$\urcorner$.
- Imagine the set of all textures and that of all fissile textures.
- Imagine the rest of the structure....

Custom categories are database schemas

$\mathcal{D} =$ 

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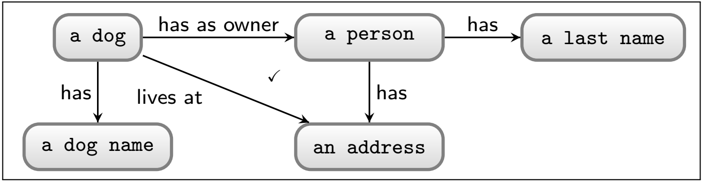

$\xrightarrow{F}$ **FinSet**

dog			
ID	**name**	**owner**	**address**
D101	Wally	P34	15 Ash St.
D102	Fido	P46	201 5th Ave.
D104	Buster	P17	27 Spring Ln.

person		
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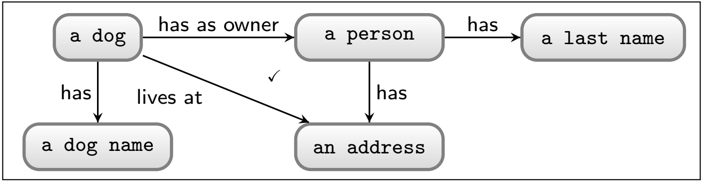
$\mathcal{D} =$  $\xrightarrow{F}$ **FinSet**

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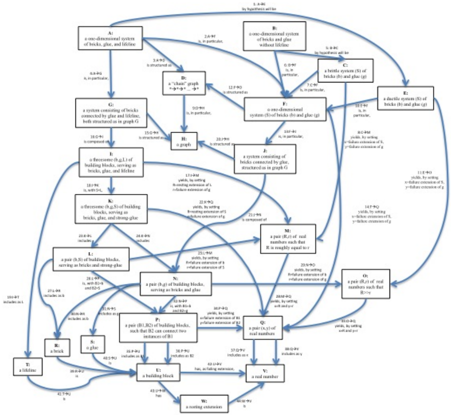
Note that equivalent paths are assigned the same composite function.

Outline

An olog from materials science

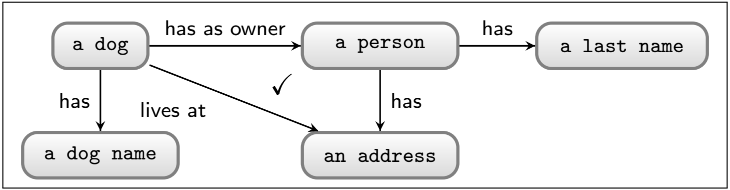
I call custom categories "ologs". Here's one from materials science.[1]



[1] Spivak, D.I.; Giesa, T.; Wood, E.; Buehler, M.J. (2011) "Category Theoretic Analysis of Hierarchical Protein Materials and Social Networks." *PLoS ONE* 6(9): e23911. doi:10.1371/journal.pone.0023911

Communicating between conceptualizations

Think of an olog as a conceptualization of something.



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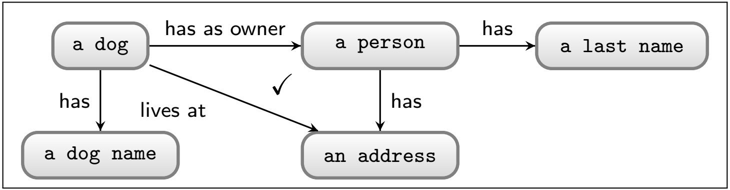

- Formally it's a category; the labels are invisible to the math.
- We can compare two ologs with a functor:
  - If you have a olog/category $\mathcal{C}$ for your conceptualization,
  - and I have a olog/category $\mathcal{D}$ for my conceptualization,
  - there may be a meaningful functor $\mathcal{C} \to \mathcal{D}$ or $\mathcal{D} \to \mathcal{C}$
  - or an overlap $\mathcal{O}$ and meaningful functors $\mathcal{C} \leftarrow \mathcal{O} \to \mathcal{D}$.

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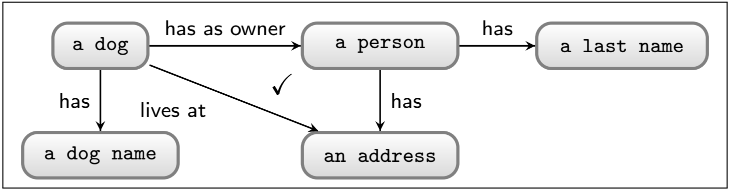

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  - or an overlap $\mathcal{O}$ and meaningful functors $\mathcal{C} \leftarrow \mathcal{O} \to \mathcal{D}$.
- But recall that filling an olog with data is also a functor $\mathcal{C} \to$ **FinSet**.

Moving data along functors

- Suppose $\mathcal{D}$ is a category and $F\colon \mathcal{D} \to \mathbf{FinSet}$ is a functor.
  - So $\mathcal{D}$ could be the dog olog and F is the data about Wally, etc.
- Suppose that $\mathcal{P}$ is another category and $i\colon P \to \mathcal{D}$ is a functor.
  - So $\mathcal{P}$ could be the person sub-olog, and i the inclusion.
- We can compose i and F to get a new functor

$$\mathcal{P} \xrightarrow{\ i\ } \mathcal{D} \xrightarrow{\ F\ } \mathbf{FinSet}, \qquad \mathcal{P} \xrightarrow{\ i;F\ } \mathbf{FinSet}$$

- So now olog $\mathcal{P}$ has data in it, borrowed from friend-olog $\mathcal{D}$ using i.

Functorial data migration

- Suppose given two conceptualizations, yours $\mathcal{C}$ and mine $\mathcal{D}$.
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- They move data from one conceptual schema to another.
- And they generalize *join, project, union* from database theory.
- So if there is even an overlap $\mathcal{C} \leftarrow \mathcal{O} \to \mathcal{D}$, we can move and share data.

Categories and functors everywhere

What to consider during the whirlwind tour:

- I'll discuss lots of different categories and functors.
- Each category is a conceptual reference frame.
  - An olog, say about dogs or shale, is a conceptual reference frame.
  - "What is the domain of discourse, and what have we agreed to?"
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Let's start the tour.

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Poset means partially ordered set $(P, \leq)$. Posets are simple categories.

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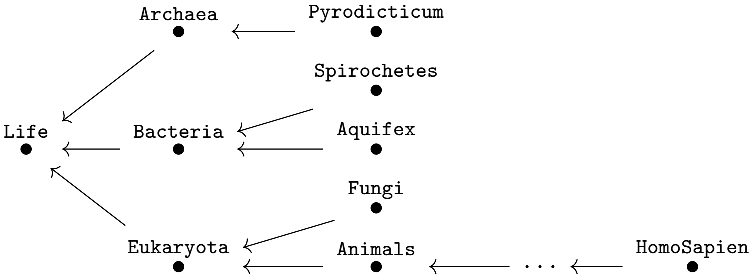
- A poset is a category with the guarantee that every two paths (having the same endpoints) are equivalent.
- For example, consider the poset $(\mathbb{R}, \leq)$ of real numbers.
  - As a category, its objects are all real numbers, $0, 1, \pi, -4.11$, etc.
  - There is (exactly one) morphism $a \to b$ iff $a \leq b$.
  - A path $a \to \cdots \to b$ just means $a \leq \cdots \leq b$, meaning $a \leq b$.

Hierarchies

- A hierarchy is just a poset, and as such is a simple category.



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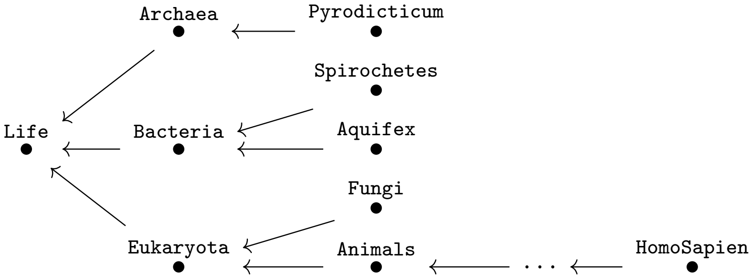

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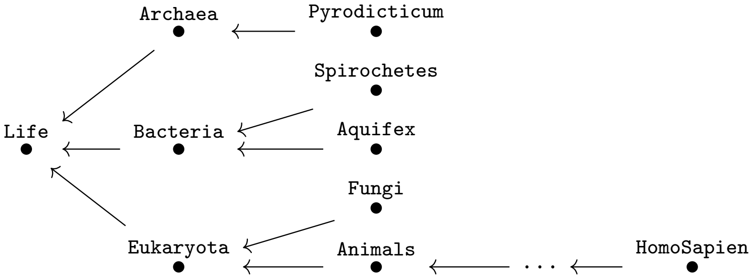

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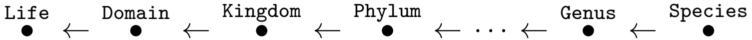
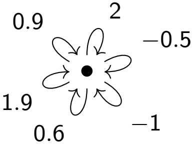


Monoids

Monoids are also simple categories, but in the opposite way.

- A *monoid* is a category with one object ("mono"=1).
- Consider the set $\mathbb{R}$ of real numbers and the multiplication operation $*$.
- These together can be considered as a category $\mathcal{R}$:
  - $\mathcal{R}$ has one object, call it $\bullet$.
  - It has an arrow $\bullet \xrightarrow{x} \bullet$ for each real number $x \in \mathbb{R}$.
  - A path is just $\bullet \xrightarrow{x_1} \cdots \xrightarrow{x_n} \bullet$.
  - We make it equivalent to its product, the one-arrow path $x_1 * \cdots * x_n$.



Actions

Consider a paint program, and let P be the set of drawable pictures.

- What actions can you take?
  - You can take any picture $p \in P$ and rotate it by an angle θ.
  - You can move any picture $p \in P$ up or down on the canvass.
  - You can fill up a closed curve in p with color.
  - Once done, you get a new picture $p' \in P$.
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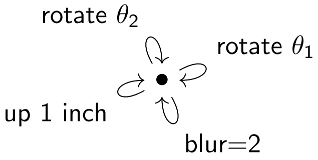
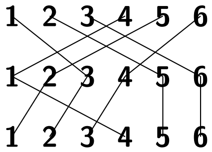
- These actions are the arrows in a monoid.
  - The "one-object" assumption says you can do any sequence of actions.
  - A path is a sequence of actions you can perform on "my pic".
  - Two paths are equal if they always act the same
  - For example rotating θ_1 then θ_2 is rotating $\theta_1 + \theta_2$.



Groups

Groups are monoids (one-object categories) where every morphism is iso.

- Recall that there are 720 isomorphisms from $\underline{6}$ to itself.
- These form a monoid: a path of isomorphisms is an isomorphism.



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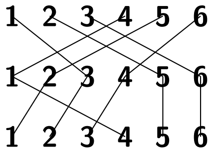

- A group is a monoid with the guarantee that every map is invertible.
- Moving a glass around is invertible, but smashing it isn't.
- Every action on a picture is invertible, except emailing it to a friend.

A group is a system of symmetries.

Sheaves

Sheaves were the original killer app for categories.

- Of course, we're talking "killer pure math app".
  - A. Grothendieck used sheaves to prove conjectures in number theory.
  - It completely transformed algebraic geometry, and other fields followed.
- Sheaves are for piecing together locally-gathered data. Examples:
  - Temperature data is gathered locally, producing worldwide coverage.
  - Security camera footage, though local, can be pieced together.
- We talk about sheaves "on a space" X, e.g. on earth's surface.

The category of all...

- One can also consider the category of all
  - finite sets,
  - groups,
  - monoids,
  - graphs,
  - posets,
  - spaces,
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  - forget structure: **Grp** $\to$ **Mon** and **Mon** $\to$ **FinSet**;
  - freely add structure: **FinSet** $\to$ **Mon** and **Mon** $\to$ **Grp**; or
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  - much more complex things like Fields medal functors $\mathbf{Shv}(X) \to \mathbf{Grp}$.
- All of math is interconnected by functors, by which we can transfer problems and solutions.

Scratching the surface

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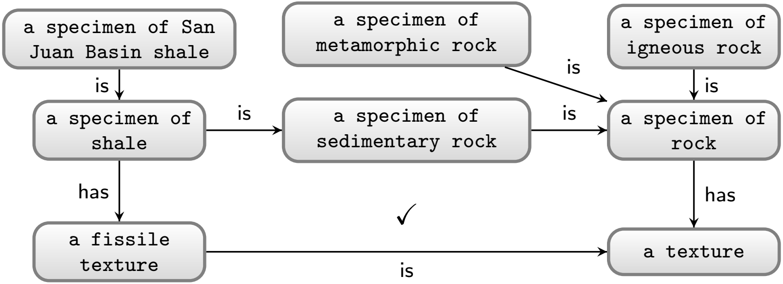

...the justice the above olog does to shale.

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What have we discussed?

- There's been a paradigm shift in math.
  - Old ideas have been categorified to show an underlying richness.
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- But the same abstractions also have applications to science.
  - A programming language is a category.
  - Custom categories can be connected by functorial bridges.
  - Actions, symmetries, hierarchies, etc. all captured by this framework.
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The future will be increasingly interdisciplinary. Category theory can help.

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- There's been a paradigm shift in math.
  - Old ideas have been categorified to show an underlying richness.
  - Subdisciplines have been connected into a rich web, using functors.
  - Different categories are fine-tuned for solving different problems.
  - Functors can transport problems and solutions between domains.
- But the same abstractions also have applications to science.
  - A programming language is a category.
  - Custom categories can be connected by functorial bridges.
  - Actions, symmetries, hierarchies, etc. all captured by this framework.
  - Data is a functor, and it can be transported by functors.

The future will be increasingly interdisciplinary. Category theory can help.

Thanks for the opportunity to speak!