Compositional analyses of dynamical systems

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Outline

E pluribus unum

Composition

Composition is assembling many things together to make one thing.[1]

[1] Color scheme and category-theoretic formalism:
"thing" = *object*, "arrangement" = *morphism*, "nesting" = *composition*.

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- One says that Y is composed of pieces $X_1, \ldots, X_n$.
  - We should specify not only the pieces, but also their arrangement.
  - We could denote an arrangement $\varphi\colon X_1, \ldots, X_n \to Y$.
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  - We could denote an arrangement $\varphi\colon X_1, \ldots, X_n \to Y$.
  - Arrangements can be nested inside each other.
- This leads naturally to the notion of operad $\mathcal{O}$, which specifies:
  - the set of possible things $X, Y, \ldots$;
  - the set of arrangements φ, ψ by which things can be composed;
  - how nesting works $\psi \circ (\varphi_1, \ldots, \varphi_n)$.

Let's give two very different examples, to see the scope.

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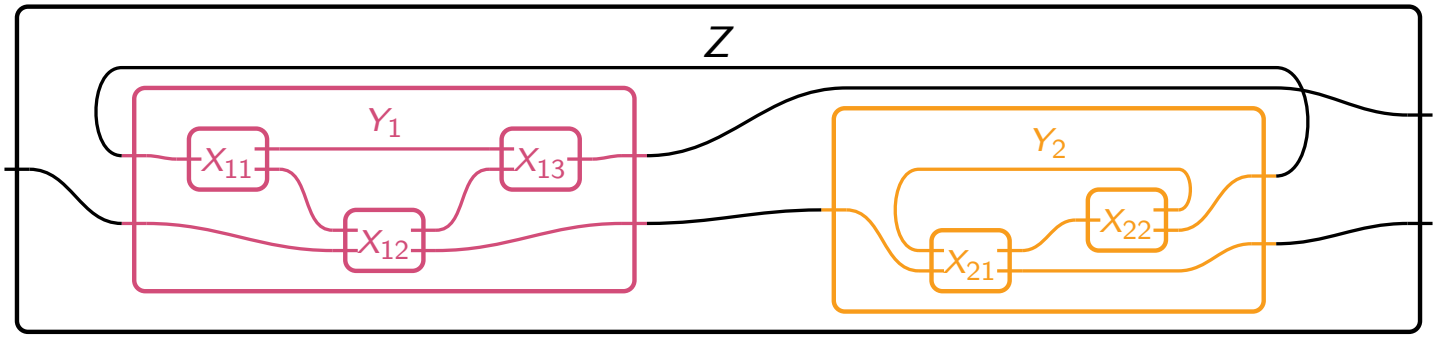
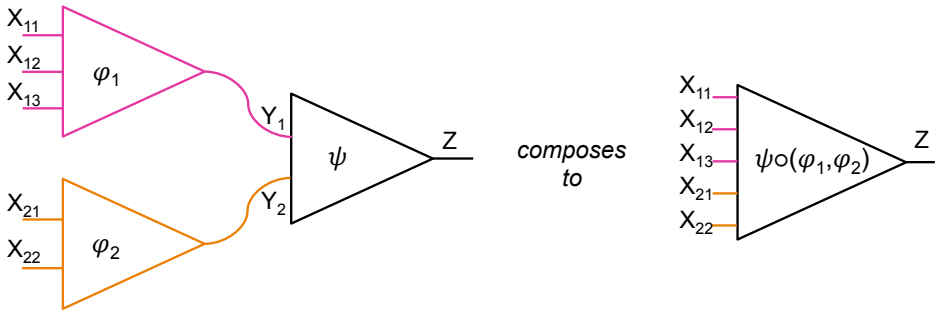
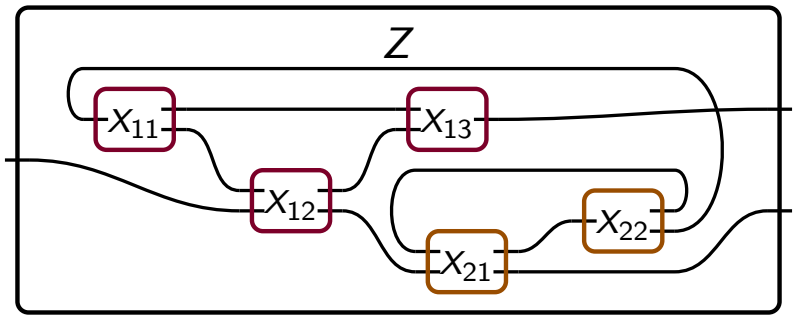
Arrangements 1: wiring diagrams

There is an operad $\mathcal{D}$ for composing hierarchical dynamical systems.

- Note the things, arrangements, and nesting here.







Arrangements 2: hierarchical protein materials

There is an operad $\mathcal{M}$ for composing hierarchical protein materials.

- A protein is an arrangement of simpler proteins.
  - There are "atomic" proteins, namely amino acids.
  - Protein materials include your skin: stretchable, breathable, waterproof.[2]

[2] Giesa, T.; Jagadeesan, R.; Spivak, D.I.; Buehler, M.J. (2015) "Matriarch: a Python library for materials architecture." *ACS Biomaterials Science & Engineering*.

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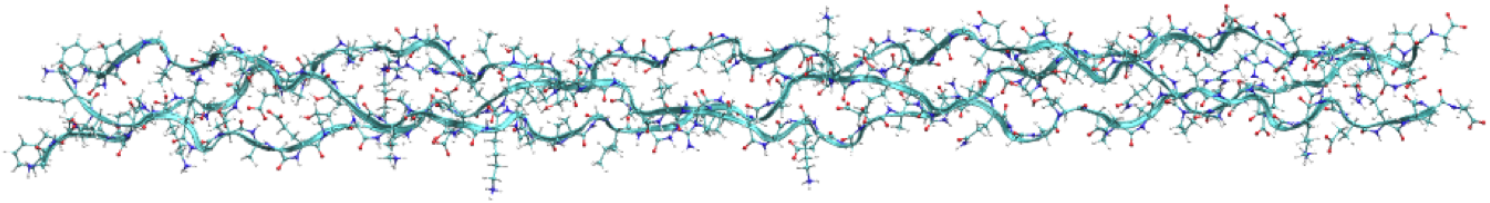
  - Protein materials include your skin: stretchable, breathable, waterproof.[2]
- A new protein can be assembled from a finite set of proteins
  - arranged in series or parallel, or
  - arranged in helices, double helices, any conceivable curve, etc.



- (Computer MD versions of) proteins model this operad.

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Plan: compositional analyses of dynamic systems

So composition—one thing out of many—can be formalized by operads.

- There's also a mathematical definition of *compositional analysis*.
  - It's something we can compute, but which also composes well:
  - Component-level analyses can be combined to provide a system-level analysis.

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  - Example: *Average* is not compositional; *sum* and *count* are.
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  - Example: *Average* is not compositional; *sum* and *count* are.
  - Always expect your machine to be used in a larger system.
- I will give several examples of compositional mappings and analyses.
  - Discretization of dynamical systems is compositional.
  - Steady state analyses are compositional.
- I'll also explain a few other compositional frameworks.
  - A new way of solving systems of non-linear equations.
  - A mathematical theory of collaborative design.

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Operads and algebras: the syntax and semantics of composition

There is a category-theoretic object, called an operad.[3]

- Operads are a framework for composing systems of systems.
- Operads are a sort of syntax; their semantics are called algebras.
- Or think operad=interface logic; algebra=inhabitants.
- Some examples of compositional structures that operads can model:
  - Every recipe—a sequence of smaller recipes—is an operad. (Performance plans might constitute an algebra for it.)

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  - Every context-free grammar is an operad.
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  - Every sort of wiring diagram, as in electric circuits or signal flow.
    (We'll discuss several different algebras for wiring diagrams.)
- I'll describe operads of wiring diagrams next.

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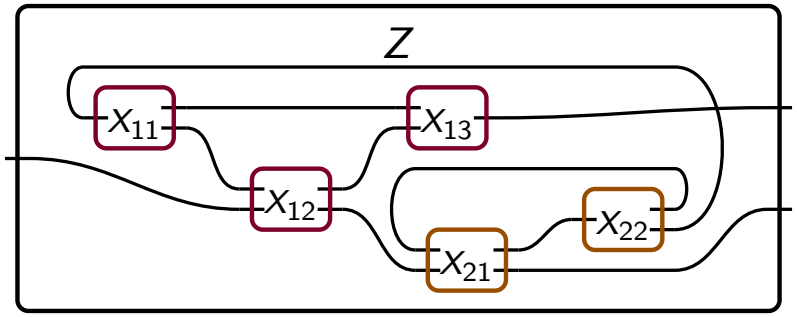
Multiple interpretations for the same operad

There is an operad of wiring diagrams.



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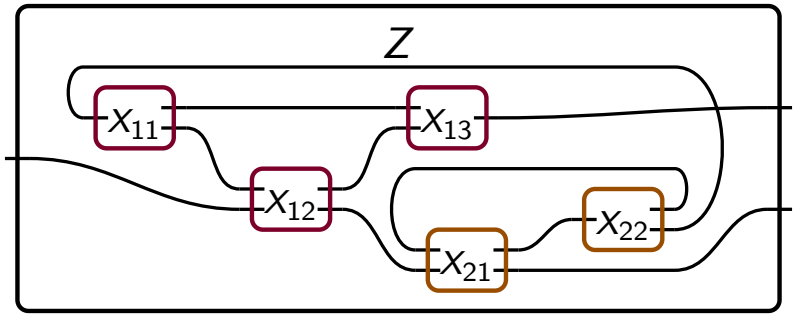

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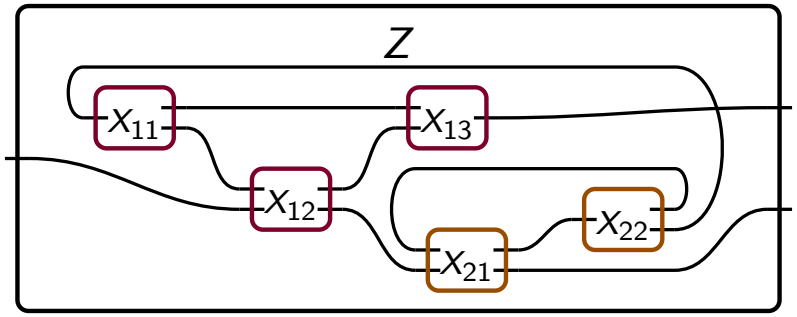

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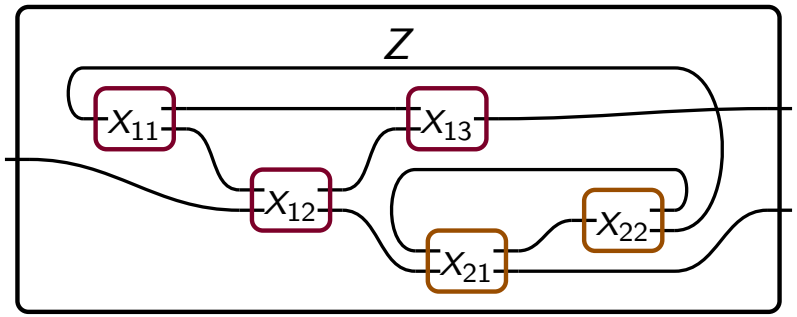
- Continuous dynamical systems compose according to these pictures.
- Discrete dynamical do too. Hybrid systems do too.
- Mathematicians recognized long ago that matrices (tensors) do too.
  - Sequential composition is "matrix multiplication".
  - Parallel composition is "Kronecker product".
  - Feedback is the block-trace operator.

Each of these semantics interpretations is formalized as an algebra.

Compositional mappings: translating between algebras

Suppose A and B are both algebras for the same operad $\mathcal{O}$.



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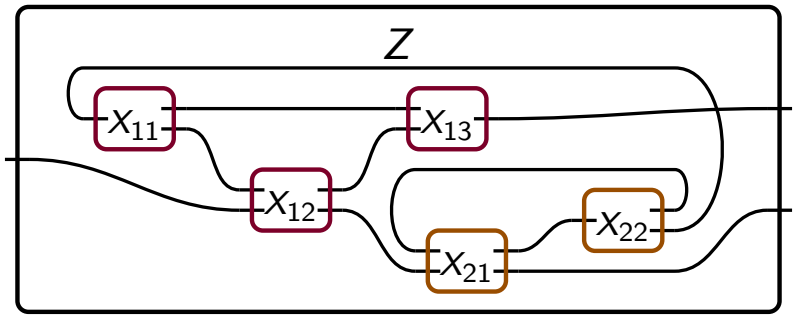

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translate-then-compose = compose-then-translate.

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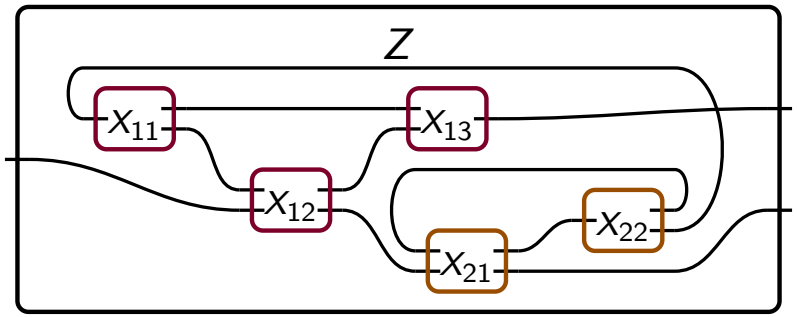

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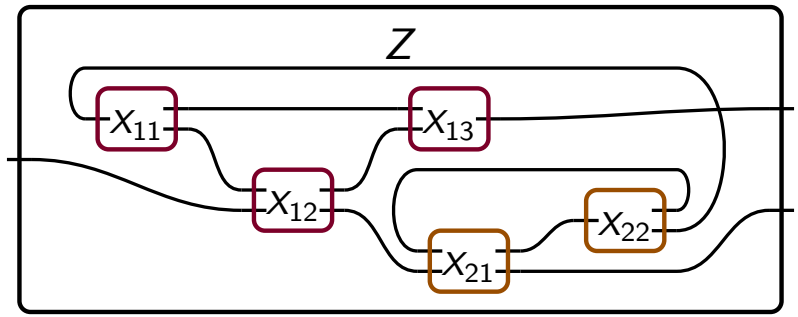
Question: is "discretizing a continuous dynamical system" compositional?

Discretization is compositional

Rephrasing the question, suppose there is a continuous DS in each box:



You have the following choices; what's the difference between them?

- Choice 1: compose the continuous systems and discretize the result;
- Choice 2: discretize each and then compose the discrete systems.

[4] Spivak "The steady states of dynamical systems". *arXiv* 1512.00802.

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Discretization is compositional

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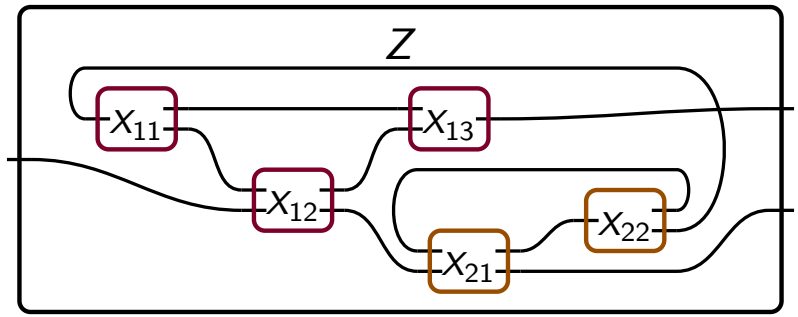

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- Choice 1: compose the continuous systems and discretize the result;
- Choice 2: discretize each and then compose the discrete systems.

"Discretization is compositional": you get the same result either way.

- This requires proof: arbitrary wiring diagrams, arbitrary DS's.
- I proved it for Euler;[4] my student proved it for Runge Kutta.[5]

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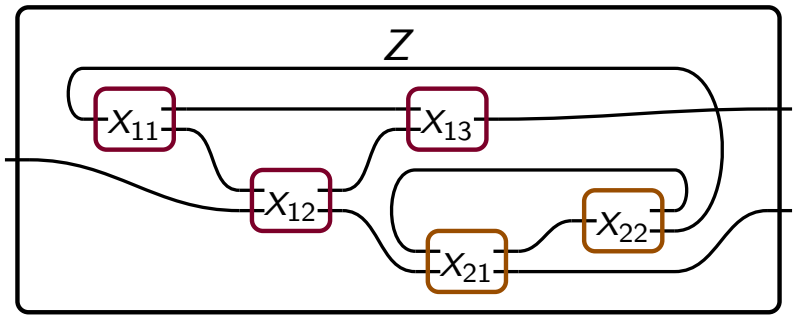

To each box, associate the "bifurcation diagram" of the DS.

- It's the n-dimensional plot of inputs/outputs that have a steady state.
- Pixelate it to form an n-dimensional tensor.
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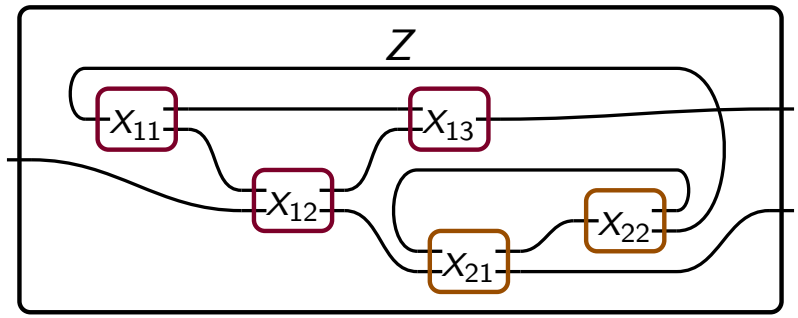

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"Bifurcation diagrams are compositional": get the same answer whether

- you first compose the DS's and then plot the bifurcation diagram; or
- you first plot the bifurcation diagrams, then compose these tensors.

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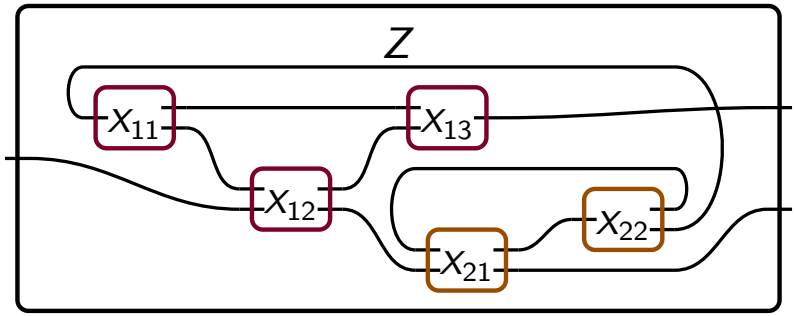
From steady states to systems of equations

Steady states of a DS are solutions to a system of equations, $\dot{\mathbf{x}} = 0$.

- For interconnected systems, the various systems are combined.



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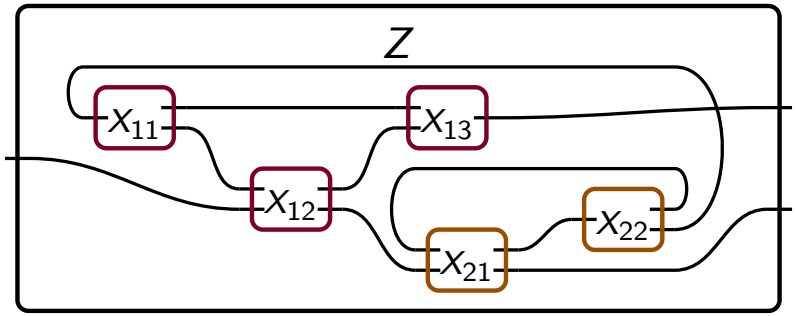

- Some variables are shared (as dictated by the wiring diagram).
- So what did "bifurcation diagrams are compositional" stuff tell us?
  - In fact, that was about plotting equations (as dots on a screen).
  - I said that to find steady states, multiply the plots as matrices.
  - Turns out this is really about solving systems of equations.
- So we're solving systems of *non-linear* equations by matrix arithmetic.

Imagining the pixel array method

Separately plot the solutions to equations: $f(x, w) = 0$ and $g(w, y) = 0$.

- Plot each in its own bounding box, say in the range $[-1.5, 1.5]$.
- Consider the plots as matrices M, N whose entries are on/off pixels.
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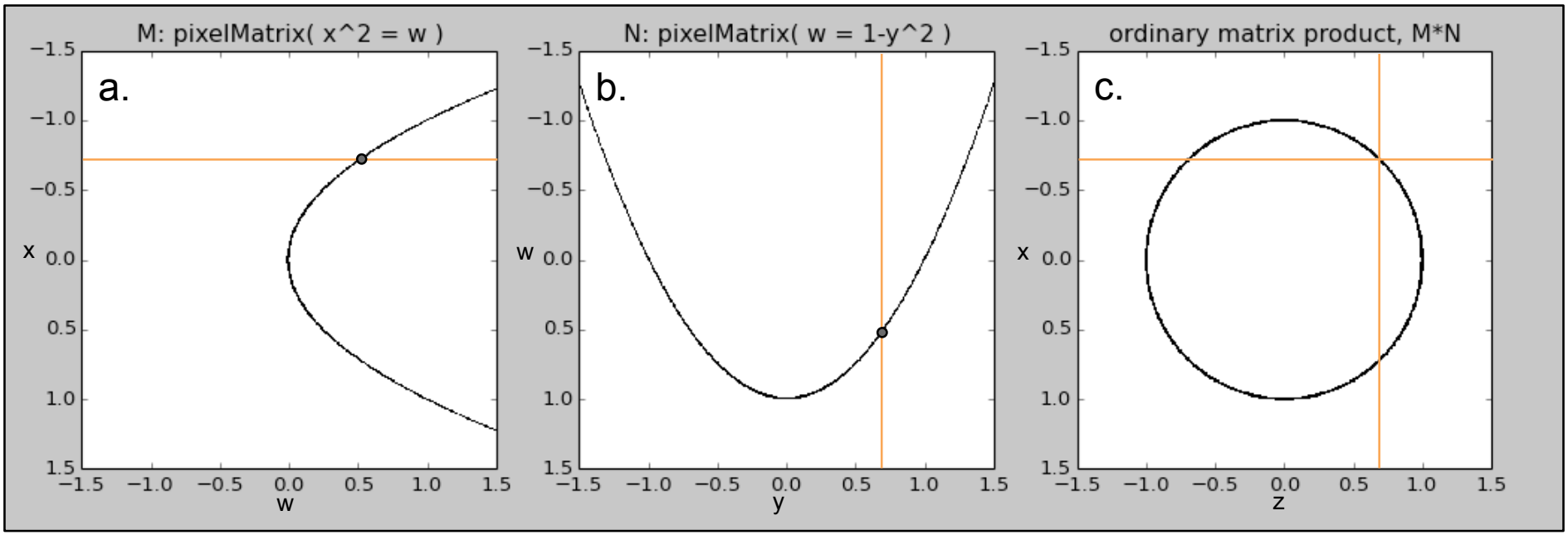
Multiplying these two matrices MN yields the simultaneous solution.

- For example, plot equations $x^2 = w$ and $w = 1 - y^2$, and multiply.



A more complex example

The following eq's are not differentiable, nor even defined everywhere.

$$\cos\left(\ln(z^2 + 10^{-3}x)\right) - x + 10^{-5}z^{-1} = 0 \qquad \text{(Equation 1)}$$

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$$\tan(x + y)(x - 2)^{-1}(x + 3)^{-1}y^{-2} = 1 \qquad \text{(Equation 3)}$$

Q: For what values of w and z does a simultaneous solution exist? [6]

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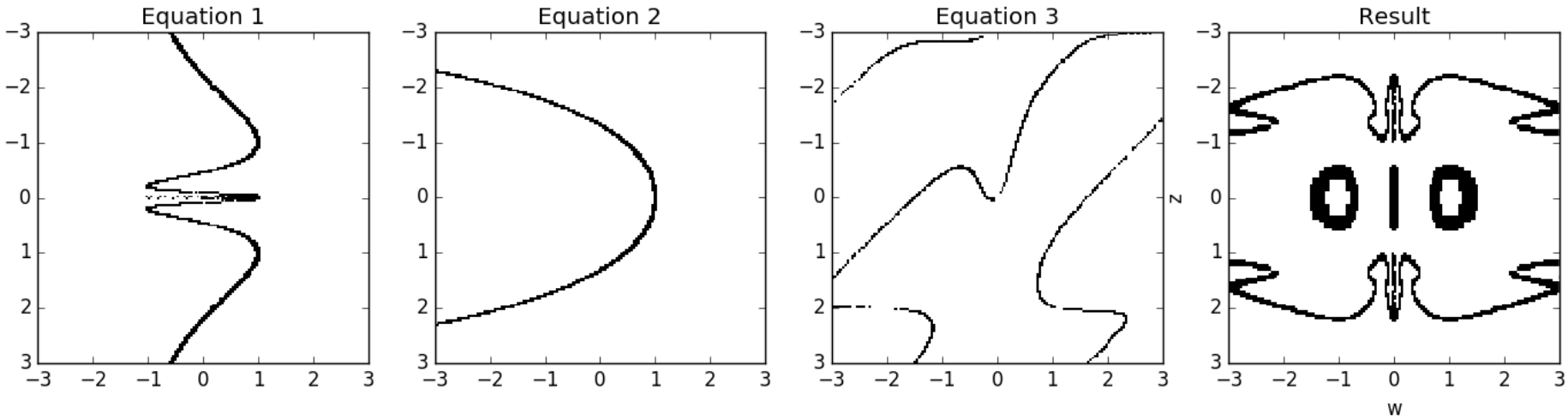

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Mathematical co-design

Designing complex systems of systems is collaborative.

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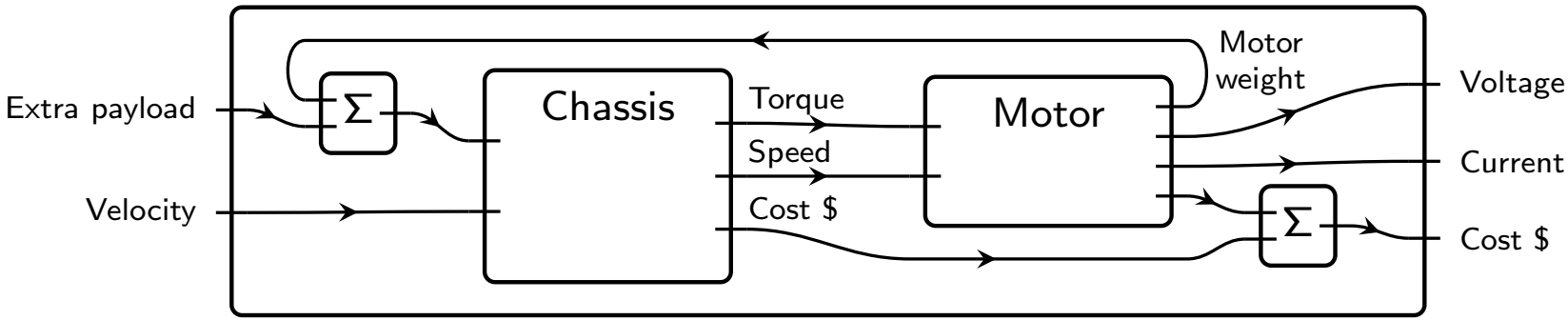

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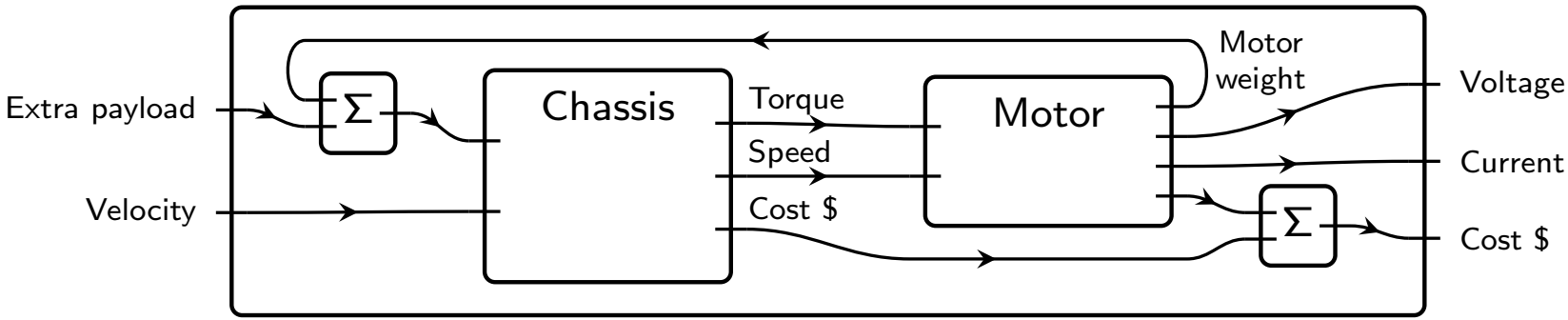
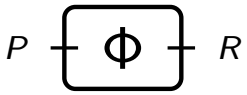


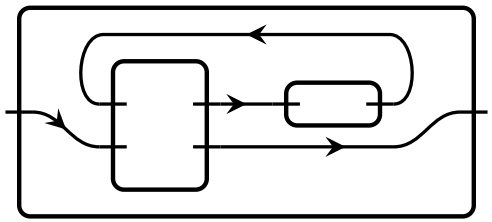
- There is a compositional theory of co-design (due to Andrea Censi).
  - Each box is a design problem.
  - Inside may be another interconnected system of design problems.

The underlying math of co-design

What is a design problem in the underlying math?



- Design is about feasibility: resources provided vs. resources required.
- Each resource (wire) is modeled as a poset, the order is by preference.
- Feasibility is a monotonic function $\Phi\colon P^{\mathrm{op}} \times R \to \{\mathtt{F}, \mathtt{T}\}$.



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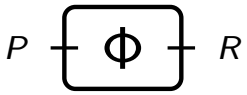
What is a design problem in the underlying math? 

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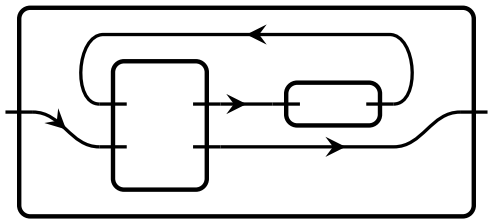
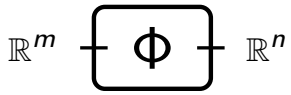


Potential application to reachability analysis for interconnected DS's.

- Imagine an initialized DS in each box. 

- To each wire, associate a poset of open sets U in the manifold.

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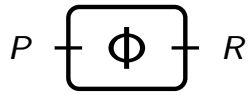
What is a design problem in the underlying math? 

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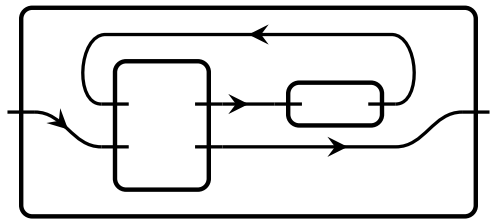
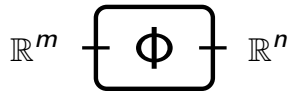


Potential application to reachability analysis for interconnected DS's.

- Imagine an initialized DS in each box. 

- To each wire, associate a poset of open sets U in the manifold.
- To a box, need $\Phi(U_1, U_2) \in \{\mathtt{F}, \mathtt{T}\}$ for each $U_1 \subseteq \mathbb{R}^m$ and $U_2 \subseteq \mathbb{R}^n$.
- Put $\Phi(U_1, U_2) = \mathtt{T}$ iff, by keeping input in U_1, output is kept in U_2.

Combining these in a system, we get the overall reachability relation.

Outline

E pluribus unum

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- We also briefly touched on operads for proteins, recipes, grammars.

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- Focused on dynamical systems; briefly also tensors and co-design.
- Compositional mappings and analyses (discretization, bifurcation).
- Discussed pixel array method for solving systems.
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Operads offer a flexible framework for discussing compositionality.

Questions and comments welcome!