The Pixel Array method for solving nonlinear systems

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Outline

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Another look at matrix multiplication

Separately plot the solutions to equations: $f(x, w) = 0$ and $g(w, y) = 0$.

- Plot each in its own bounding box, say in the range $[-1.5, 1.5]$.
- Consider the plots as matrices M, N whose entries are on/off pixels.
- That is, M and N are *boolean matrices* corresponding to f and g.

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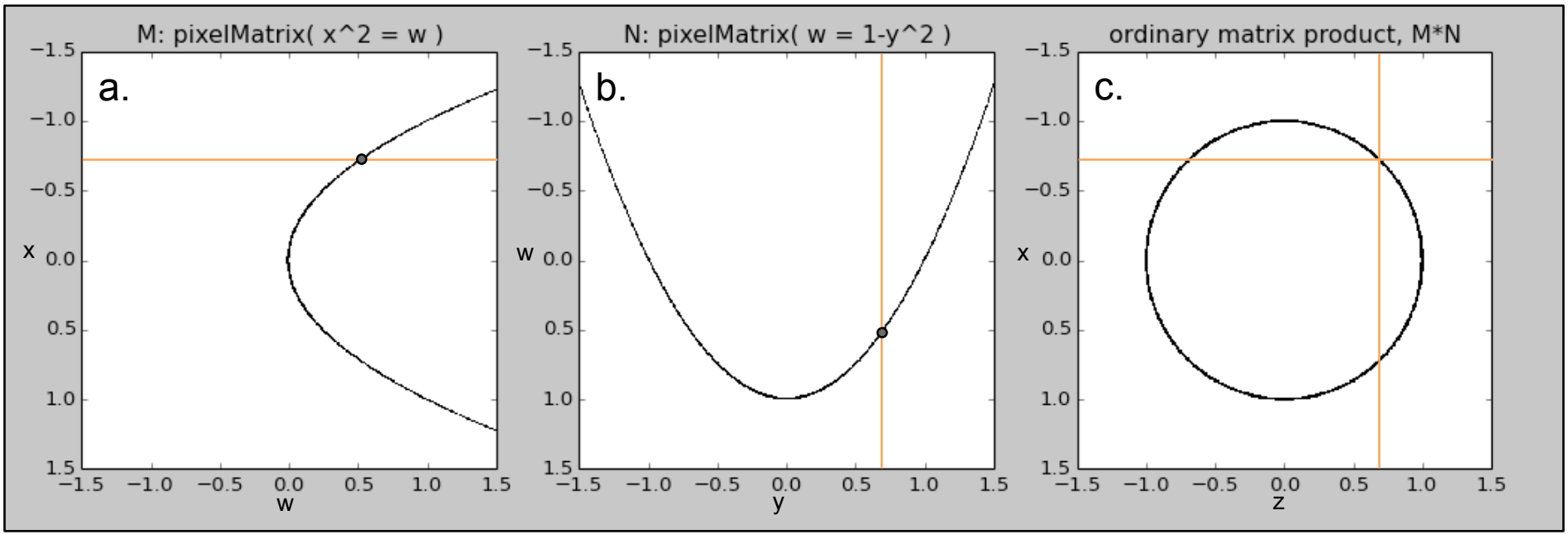
Multiplying these two matrices MN yields the simultaneous solution.

- For example, plot equations $x^2 = w$ and $w = 1 - y^2$, and multiply.



A more complex example

The following eq's are not differentiable, nor even defined everywhere.

$$\cos\left(\ln(z^2 + 10^{-3}x)\right) - x + 10^{-5}z^{-1} = 0 \qquad \text{(Equation 1)}$$

$$\cosh(w + 10^{-3}y) + y + 10^{-4}w = 2 \qquad \text{(Equation 2)}$$

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Q: For what values of w and z does a simultaneous solution exist? [1]

[1] Spivak, DI; Dobson, MRC; Kumari, S. (2016) "Pixel Arrays: A fast and elementary method for solving nonlinear systems". http://arxiv.org/pdf/1609.00061v1.pdf

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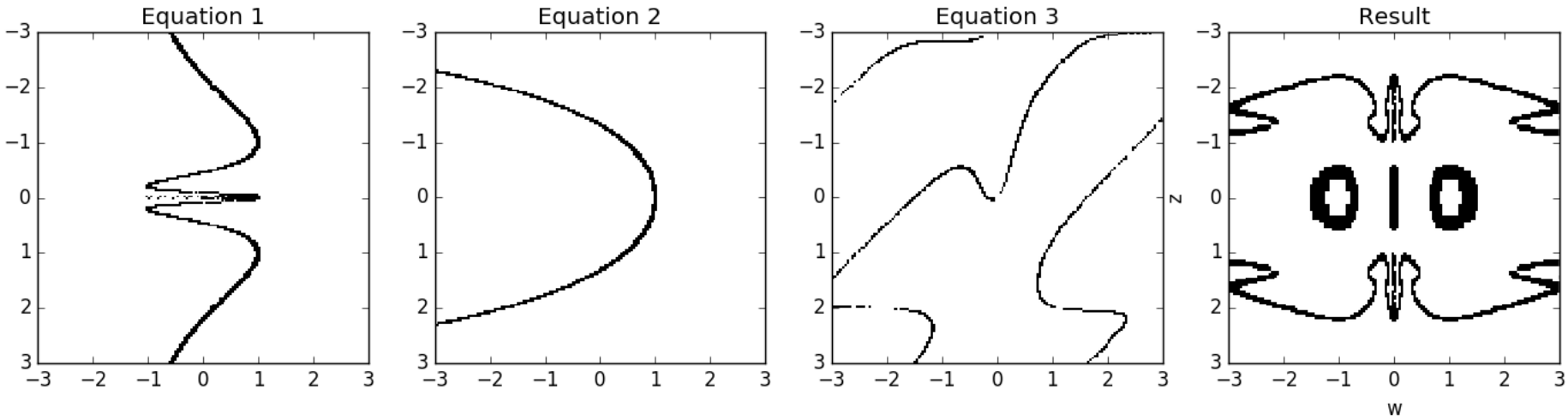

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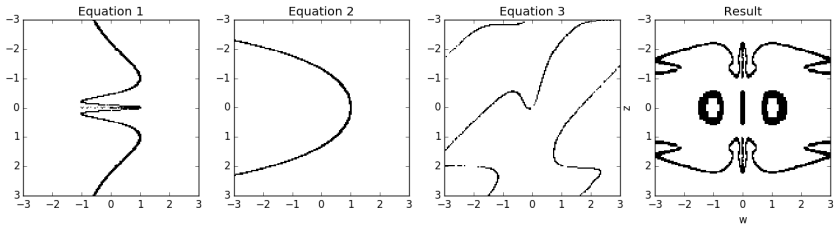
Selling points

The Pixel Array method has the following features:

- it returns *all solutions in a given bounding box*;
- it's *much faster* than quasi-Newton methods for finding "all solutions" to partially decomposable systems;
- it introduces *no false negatives*;
- it works for *non-differentiable* or even *discontinuous* functions;
- it's *not iterative* and requires no initial guess, in contrast with quasi-Newton methods;
- it's *elementary*—just clustered tensor multiplication—hence has low barrier to entry; and
- it *provides insights*, by showing the whole solution set at once.



So where are the limitations hiding?

Limitation: PA method takes original plots as input.

- The Pixel Array (PA) method requires a plot for each equation.
- The plot of $f(x_1, \ldots, x_n) = 0$ is n-dimensional; can be costly.
  - But there are methods: sampling, zero-crossing, "interval arithmetic".
  - Nice aside: you can even use raw data as your plots.
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Limitation: its speed requires unexposed variables, partial decomposability.

- I will discuss exposed/unexposed variables a bit more.
- For now, recall butterfly picture: we exposed w and z, but not x or y.
- Speed comparison with quasi-Newton is not straightforward.

Outline

Equations and wiring diagrams

Consider an arbitrary system of equations having the following form:

$$f_1(\mathbf{t}, u, \mathbf{v}) = 0$$
$$f_2(\mathbf{v}, w, x) = 0$$
$$f_3(u, w, x, y) = 0$$
$$f_4(x, \mathbf{z}) = 0$$

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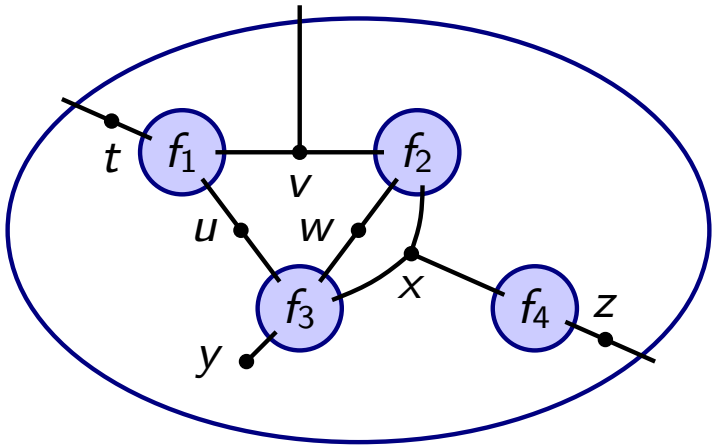

Said another way, we want $\{(t, v, z) \mid \exists u, w, x, y : f_1 = f_2 = f_3 = f_4 = 0\}$.

Using array multiplication to solve systems

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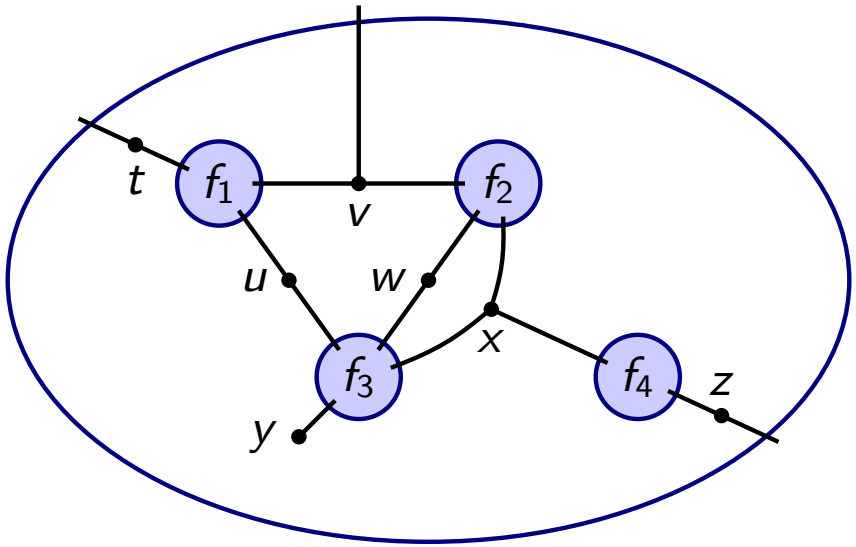
- To solve the system for t, v, z, we plot each equation as an array.
- PA says: multiply the arrays according to "variable sharing" diagram:



Partial decomposability: not every equation shares every variable

Why and how

There are now two important directions to go from here:

- Explain general array multiplication as dictated by a wiring diagram.
- Demystify relationship between multiplying arrays and solving systems.

Let's start by demystifying why the Pixel Array method works.

- As we'll see, it just comes down to logic: AND, OR, TRUE, FALSE.

To keep things simple, let's restrict our attention to matrix multiplication.

Multiplying Boolean matrices

The matrix multiplication formula works well in any semiring.

$$(MN)_{i,k} = \sum_j M_{i,j} * N_{j,k}.$$

- Roughly, a semiring is a set with $0, 1, +, *$ that act reasonably.
- It's like a ring, but you don't need negatives (e.g. $\mathbb{N}$).

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Today we'll focus on the Boolean semiring, $\mathbb{B}$.

- It has two elements $\mathbb{B} = \{0, 1\}$.
- 0 means FALSE, and 1 means TRUE.
- Multiplication is given by boolean AND (denoted $\wedge$).
- Addition is given by boolean OR (denoted $\vee$).
- The only slightly unexpected thing is that $1 + 1 = 1$.

The logic of matrix multiplication

In general, the PA method is to multiply boolean tensors (arrays).

- But for now, let's suppose A, B are boolean matrices.
- Say that A and B are plots of $f(x, y) = 0$ and $g(y, z) = 0$, resp.
- So $A_{i,j} = 1$ means $f(x, y) = 0$ in corresponding pixel (else $A_{i,j} = 0$).

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- The (i, k)-entry of their product AB is given by the formula:

$$\begin{aligned}(AB)_{i,k} &= \sum_j A_{i,j} * B_{j,k} \\ &= \bigvee_j A_{i,j} \wedge B_{j,k} \\ &= \exists_j \left(A_{i,j} \wedge B_{j,k}\right)\end{aligned}$$

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- " There exists some y such that $f(x, y) = 0$ and $g(y, z) = 0$. "

Multiplying larger-order arrays

When two arrays share a common dimension, they can be multiplied.

- For example, say that A is $r_1 \times \cdots \times r_m$ and that B is $s_1 \times \cdots \times s_n$.
- If $r_1 = s_1$, the product is an $r_2 \times \cdots \times r_m \times s_2 \times \cdots \times s_n$ array.

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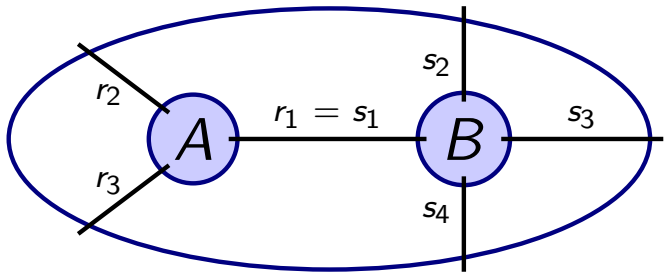
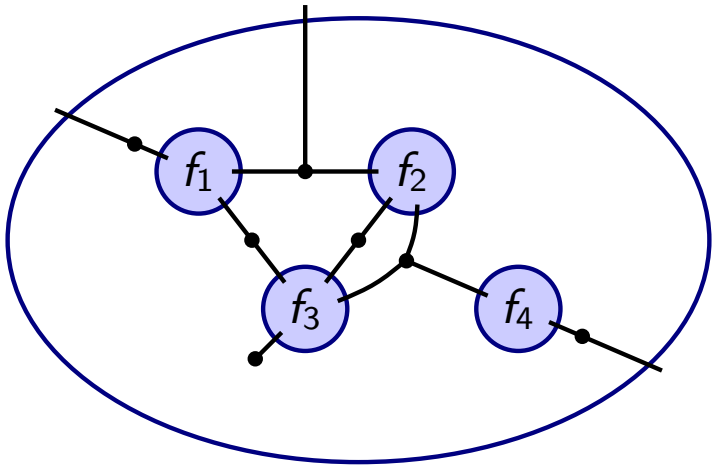
We will be drawing these situations using wiring diagrams.



The general array multiplication algorithm



A single formula exists to multiply arrays according to any wiring diagram.

- Basically: iterate over all the links, multiply array entries, and sum.

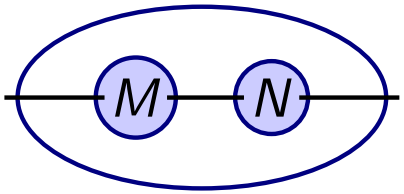
$$\text{Result}(\text{exposed\_vars}) = \sum_{\text{unexposed\_vars}} \prod_{\text{tensors}} \text{tensor}(\text{relevant\_vars})$$

- But this is very naive: $O(\text{resolution}^{\#\text{vars}})$.
  - Plots of equations are sparse, boolean matrices (entries are bits, 0,1).
  - Both of sparsity and booleanness can speed up matrix multiplication.
- But to be efficient, we will need to "cluster" the multiplication.

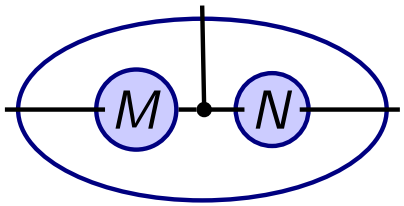
Example wiring diagrams for named operations

The same general array multiplication formula[2] returns famous matrix products for the following wiring diagrams:

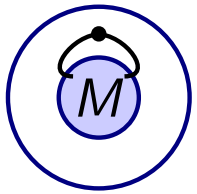
Multiplication: MN



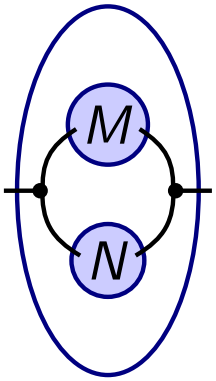
Khatri-Rao: $M \odot N$



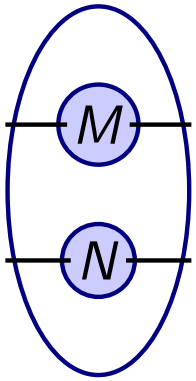
Trace: $\mathrm{Tr}(M)$



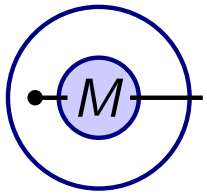
Hadamard: $M \circ N$



Kronecker: $M \otimes N$



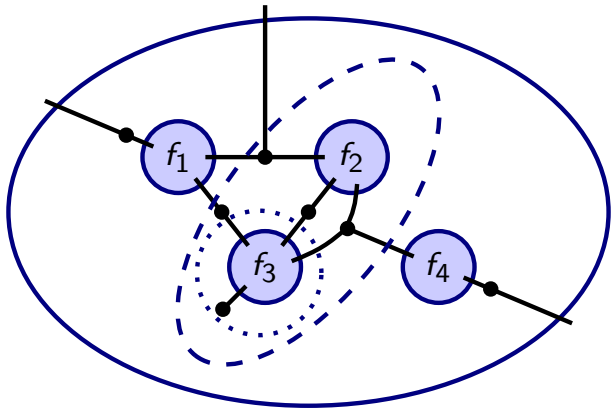
Marginalize: $\sum_i M_{i,j}$



[2] $\text{Result}(\text{exposed\_vars}) = \sum_{\text{unexposed\_vars}} \prod_{\text{tensors}} \text{tensor}(\text{relevant\_vars})$

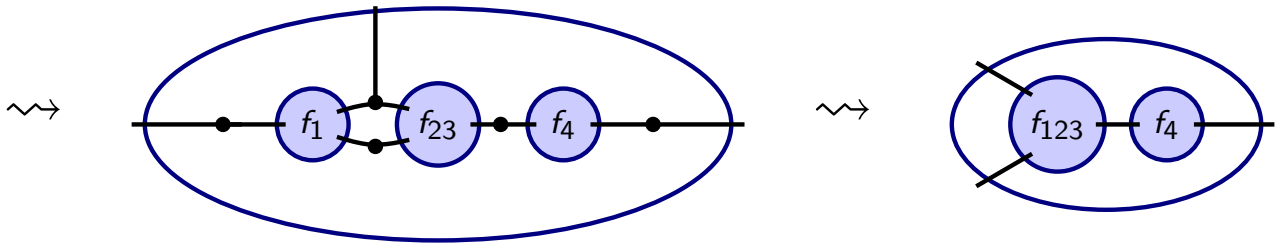
Clustering for speed

Order of array multiplication doesn't affect solution, but does affect speed.



Naive cost for multiplying arrays: $O(\text{resolution}^{\#\text{vars}})$

- Even if we can use sparsity, etc. solving big systems is expensive.
- Instead, we should use the associative law: cluster.



- Naive cost goes from $O(\text{resolution}^7)$ to $O(\text{resolution}^4)$.

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Apples and oranges

The inputs and outputs of the PA method are different than Newton's.

	Pixel Array method	**Newton's method**
Inputs:	a range for each variable	a good initial guess
Outputs:	all solutions for some variables	one solution in all variables.

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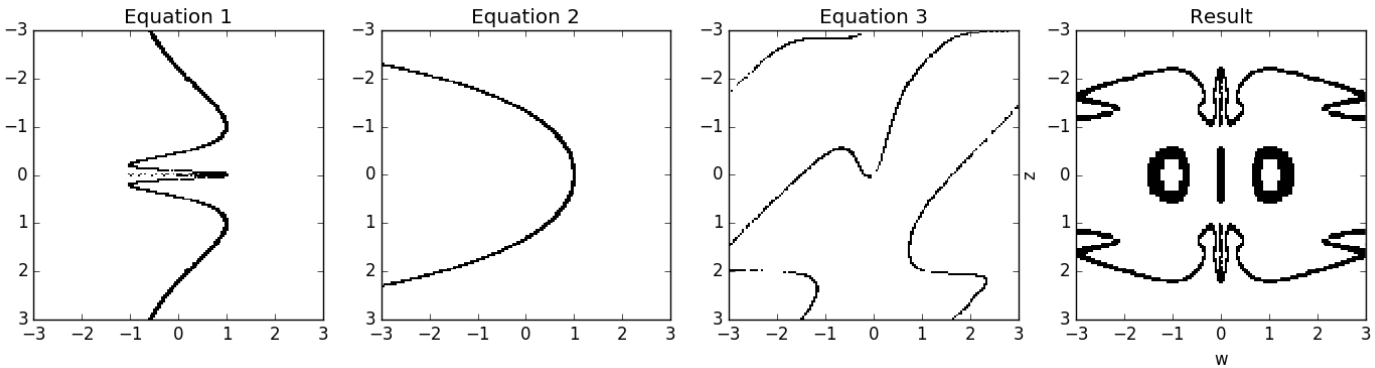
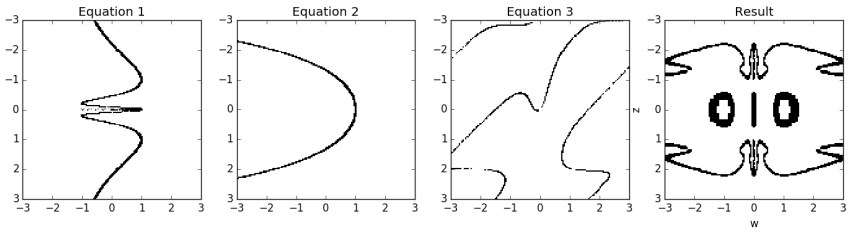
Our speed test assumes you want to produce "all solutions" in range.



How might we use Newton to find all solutions??

How to compare speed?



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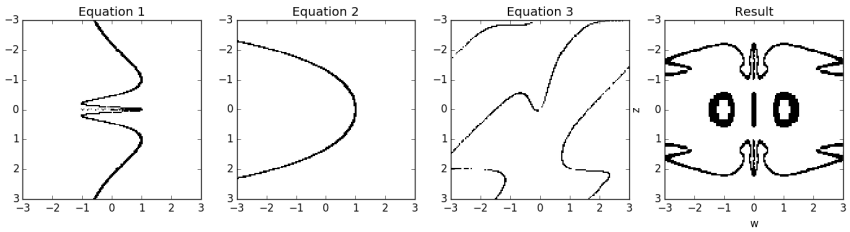

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For the above case, PA was over 7200x faster in finding all solutions.

- PA: 1.5 seconds; Newton: we stopped Julia's NLsolve after 3 hours.
- PA was faster in every partially-decomposable system we tried.

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In this talk, I discussed the Pixel Array method for solving systems.

- We have an array (tensor) for each equation.
- We know which variables are shared (captured by a wiring diagram).
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Thanks for your time;
Questions and comments welcome!