

# Language and semantics for behavior

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## I. Introduction

### A. Behavior over time.

1. Labeled transition systems
2. Differential equations
3. Delays

### B. Two problems with $\text{Shv}(\mathbb{R})$

1. Non-composability
  - a. Example: roughly monotonic.
2. Translation invariance
  - a. Things can depend on Time, but it is explicit.

## II. The domain $\mathbb{IR}$

A. As a poset  $[d, u] \sqsubseteq [d', u']$  iff  $d \leq d' \leq u' \leq u$ .

B. The way-below relation  $[d, u] \ll [d', u']$  iff  $d < d' \leq u' < u$ .

### C. The Scott topology

1. Basis
2. Bijection to half-plane
3. Open sets as Lipschitz functions

## III. The continuous category $S_{\mathbb{IR}/\triangleright}$

A. Action of  $\mathbb{R}$  on  $\mathbb{IR}$ .

### B. Quotient

1. ...as Grothendieck construction
2.  $\mathbb{IR}/\triangleright$ 
  - a. Objects  $\ell$ ,

- b. morphisms  $\langle r, s \rangle: \ell' \rightarrow \ell$ ,
- c. composition by  $\langle +, + \rangle$ .

C. As a site

- 1. Wavy arrows  $\langle r, s \rangle: \ell' \rightsquigarrow \ell$  for  $r, s > 0$ .
- 2. One covering family of  $\ell$ : the wavy arrows to  $\ell' \rightsquigarrow \ell$ .

D. The topos  $\mathcal{B} = \text{Shv}(S_{\mathbb{R}/\triangleright})$ .

- 1. Time as a sheaf
- 2.  $\mathcal{B}$  as Étendue:  $\mathcal{B}/\text{Time} \cong \text{Shv}(\mathbb{R})$ .
- 3. Subobject classifier: Dyck paths.

IV. Internal language

A. Signature: Types, terms, axioms

- 1. No new types, one new term, many axioms.

B. Modalities in general

- 1. Internal monad.
- 2. TFAE  $(P \rightarrow Q) \Rightarrow jP \Rightarrow jQ$ ,  $j(P \wedge Q) \Leftrightarrow (jP \wedge jQ)$
- 3. The order  $\leq$
- 4. Open, closed, quasi-closed

C. In our case:  $\hat{\uparrow}_{[d,u]'} \downarrow_{[d,u]'} @_{[d,u]'} \pi$ .

- 1. Dyck path picture
- 2. Subtopos idea