

Cayley graphs and the Grothendieck construction

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Abstract

Category theory is the gateway to learning pure mathematics. It organizes concepts that appear throughout mathematics into a coherent framework and highlights commonalities that appear throughout.

The Grothendieck construction is one example. It generalizes two important ideas from group theory: the Cayley graph of a group and the semi-direct product of groups. Because of time constraints, I only have time to talk about one of these, namely Cayley graphs; explaining how the Grothendieck construction generalizes semi-direct products would require an additional 8 minutes (so stick around and ask me afterwards!)

I'll assume the audience knows what groups and group homomorphisms are. But I will define categories, functors, the Yoneda functor, the Grothendieck construction, and the Cayley graph of a group during the talk.

1 Groups and Cayley graphs

1.1 Groups

A group is a monoid with inverses. Here's the formal definition.

Definition 1.1 (Monoid, Group). A *monoid* consists of three components $(G, \text{id}, *)$, where

1. G is a set, called the *carrier*,
2. $\text{id} \in G$ is an element, called the *identity element*, and
3. $*$: $G \times G \rightarrow G$ is a function, called *the multiplication function*.

We denote $*(g_1, g_2)$ by $g_1 * g_2$. The above components must satisfy two conditions:

- i. $\text{id} * g = g = g * \text{id}$, for all $g \in G$ and
- ii. $(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$, for all $g_1, g_2, g_3 \in G$.

Condition i. is called the *unital law*, and Condition ii. is called the *associative law*. If the monoid furthermore satisfies the condition

$$\text{iii. for all } g \in G \text{ there exists a } h \in G, \text{ such that } g * h = \text{id} = h * g$$

then we say that G is a group. Condition iii. is called *existence of inverses*.

Proposition 1.2. *Suppose $(G, \text{id}, *)$ is a group. Then for every $g \in G$ there is exactly one g' such that $g * g' = \text{id}$.*

Proof. By the inverse law, we already have an element $h \in G$ such that $g * h = \text{id} = h * g$. So if g' is any element satisfying $g * g' = \text{id}$, then we can multiply both sides by h and, using the unital law and the associative law, obtain

$$g' = \text{id} * g' = (h * g) * g' = h * (g * g') = h * \text{id} = h.$$

So g' and h are the same. □

Because of Proposition 1.2, elements in groups have *unique inverses*. Thus for every $g \in G$ there is a unique element h such that $g * h = \text{id}$; we can denote it g^{-1} . It automatically satisfies $g * g^{-1} = \text{id} = g^{-1} * g$. One might say “a group is a monoid in which every element is invertible.”

Example 1.3. The dihedral group D_8 is the set of symmetries of a square. Let ρ stands for rotate, and ϕ stands for flip horizontally. These satisfy the property that $\rho^4 = \text{id}$ and $\phi^2 = \text{id}$ because rotating a square four times or flipping it horizontally twice returns it to the original position. But so does rotate-flip-rotate-flip, so $\rho\phi\rho\phi = \text{id}$. The group is said to be *presented by generators* $\{\rho, \phi\}$ *and relations* $\{\rho^4 = \text{id}, \phi^2 = \text{id}, \rho\phi\rho\phi = \text{id}\}$.

One can check that the carrier of D_8 consists of eight elements:

$$\{\text{id}, \rho, \rho * \rho, \rho * \rho * \rho, \phi, \rho * \phi, \rho * \rho * \phi, \rho * \rho * \rho * \phi\}.$$

Exercise: prove the following: $\rho\phi\rho = \phi$ and $\phi\rho = \rho^3\phi$ and $\rho^2\phi\rho = \rho\phi$.

1.2 Cayley graphs

To every monoid, there is an associated Cayley graph. Groups being monoids, they too have Cayley graphs. But before we say what these are, we should say what a graph is.

Definition 1.4 (Graph). A graph consists of four components (V, E, s, t) , where

1. V is a set, elements of which are called *vertices*,
2. E is a set, elements of which are called *edges*,
3. $s: E \rightarrow V$ is a function, called the *source* function, and
4. $t: E \rightarrow V$ is a function, called the *target* function.

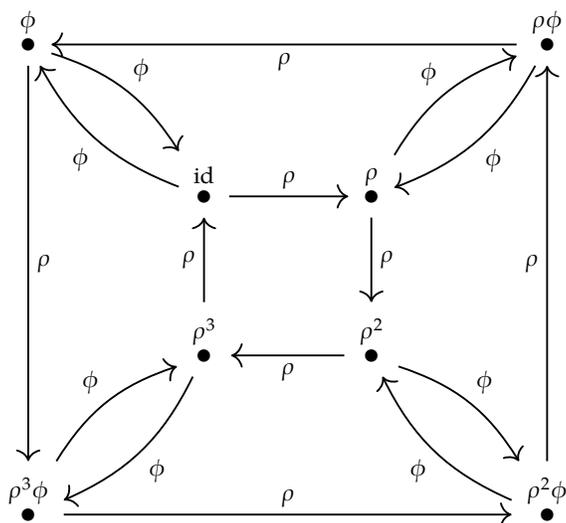
Given a graph (V, E, s, t) , a *path of length n* (for $n = 0, 1, 2, \dots$) is a set of vertices $\{v_0, \dots, v_n\}$ and edges $\{e_1, \dots, e_n\}$ such that $s(e_i) = v_{i-1}$ and $t(e_i) = v_i$. The vertex v_0 is called the *source* of the path and the vertex v_n is called the *target* of the path.

Example 1.5. Consider the graph with one vertex $V = \{\cdot\}$, and two edges $\{\phi, \rho\}$. Note that every path through this graph corresponds to an element in D_8 .

Definition 1.6 (Cayley graph). Let $(G, \text{id}, *)$ be a monoid and $A = \{a_1, \dots, a_n\}$ a set of generators. The *associated Cayley graph* has

1. the set of vertices is $V := G$,
2. the set of edges is $E := G \times A$,
3. the source function is $s(g, a) = g$,
4. the target function is $t(g, a) = g * a$.

Example 1.7. The Cayley graph of D_8 , with generators $\{\rho, \phi\}$ is shown below



See Example 1.3.

2 Categories and Functors

2.1 Categories

Definition 2.1 (Category). A *category C* consists of four components:

1. a collection $\text{Ob}(C)$, each elements of which is called an *object*;
2. for every $c_1, c_2 \in \text{Ob}(C)$, a set $\text{Mor}_C(c_1, c_2)$, each element f of which is called a *morphism from c_1 to c_2* and denoted $f: c_1 \rightarrow c_2$ or $c_1 \xrightarrow{f} c_2$. The *domain* of f is c_1 and the *codomain* of f is c_2 ;

3. for every $c \in \text{Ob}(C)$, a chosen morphism $\text{id}_c \in \text{Mor}_C(c, c)$, called the *identity morphism on c* ; and
4. for every three objects $c_1, c_2, c_3 \in \text{Ob}(C)$, a function $*$: $\text{Mor}_C(c_1, c_2) \times \text{Mor}_C(c_2, c_3) \rightarrow \text{Mor}_C(c_1, c_3)$, called the *composition formula*. Given morphisms $f: c_1 \rightarrow c_2$ and $g: c_2 \rightarrow c_3$, we denote the image of (f, g) by $f * g$.

These components are required to satisfy two properties:

- i. for every $f: c_1 \rightarrow c_2$, there are equations $\text{id}_{c_1} * f = f = f * \text{id}_{c_2}$ and
- ii. for every $c_1 \xrightarrow{f} c_2 \xrightarrow{g} c_3 \xrightarrow{h} c_4$, there is an equation $f * (g * h) = (f * g) * h$.

The first is called the *unital law* and the second is called the *associative law*.

Example 2.2 (The category of sets). The category of sets, denoted **Set**, has the collection of all sets (in a given universe) as its objects. Given sets S_1 and S_2 , the set $\text{Mor}_{\text{Set}}(S_1, S_2)$ is defined to be the set of functions from S_1 to S_2 ; in other words, a morphism $f: S_1 \rightarrow S_2$ in **Set** is just a function. The identity for any set is the usual identity function, and the composition of two functions is the usual composition.

Example 2.3 (The category of groups). The category of groups, denoted **Grp**, has the collection of all groups (in a given universe) as its objects. Given groups G_1 and G_2 , the set $\text{Mor}_{\text{Grp}}(G_1, G_2)$ is the set of groups homomorphisms. The identity morphism and composition of morphisms are as usual.

Similar for monoids.

Example 2.4 (Categories with one object). If C is a category with one object, say $\text{Ob}(C) = \{\cdot\}$, then Definition 2.1 gets simpler. Let $M := \text{Mor}_C(\cdot, \cdot)$. The identity becomes an element $\text{id} \in M$ and the composition formula becomes a function $*$: $M \times M \rightarrow M$. Look familiar? It's just a monoid in disguise!

Remark 2.5. Just like you can present a group by generators and relations, you can present a category C by generators and relations. It's sometimes called a *linear sketch*. You say what all the objects are, and then you give a set $A \subseteq \text{Mor}(C)$ of morphisms, called *generating morphisms*. From here you get a graph. Finally, you give some equations between paths in the graph, called *relations*. If two paths are declared equal, they must have the same source and they must have the same target.

Each category-with-generators (C, A) has an underlying graph, whose vertices are the objects of C and whose edges are the elements of A .

Definition 2.6 (Isomorphism). Let C be a category. A morphism $f: c \rightarrow d$ is called an *isomorphism* if there exists a morphism $g: d \rightarrow c$ such that $f * g = \text{id}_c$ and $g * f = \text{id}_d$. In this case we say that f is *invertible* and that its *inverse* is g . An *automorphism* is an isomorphism $c \rightarrow c$ for some object c .

A category in which every morphism is an isomorphism is called a *groupoid*.

Example 2.7. An isomorphism in the category **Set** turns out to be the same as a bijection, i.e. a function that is one-to-one and onto. For example, the sets $\{a, b, c\}$ and $\{1, 2, 3\}$ are isomorphic. In fact, there are six (3 factorial) different isomorphisms, and they form a group.

Example 2.8 (Groupoid with one object). Just like a category with one object is a monoid, a groupoid with one object is a group. The naming convention is unfortunate, but *c'est la vie*.

The idea is easy: a groupoid with one object is in particular a category with one object, so it's just a monoid in disguise. Add the fact that every morphism is invertible, and you get a group by definition.

2.2 Functors

Definition 2.9 (Functor). Let \mathcal{C} and \mathcal{D} be categories. A *functor* $F: \mathcal{C} \rightarrow \mathcal{D}$ consists of two components:

1. a function $\text{Ob}(F): \text{Ob}(\mathcal{C}) \rightarrow \text{Ob}(\mathcal{D})$, called the *on-objects part* and often denoted simply F , and
2. for every $c_1, c_2 \in \text{Ob}(\mathcal{C})$, a function $\text{Mor}_F(c_1, c_2): \text{Mor}_{\mathcal{C}}(c_1, c_2) \rightarrow \text{Mor}_{\mathcal{D}}(F(c_1), F(c_2))$.

Often $\text{Ob}(F)$ and $\text{Mor}_F(c_1, c_2)$ are denoted simply by F . These components are required to satisfy two conditions:

- i. for all $c \in \text{Ob}(\mathcal{C})$, an equation $F(\text{id}_c) = \text{id}_{F(c)}$,
- ii. for all $c_1 \xrightarrow{f_1} c_2 \xrightarrow{f_2} c_3$, an equation $F(f_1 * f_2) = F(f_1) * F(f_2)$.

Example 2.10 (Grp \rightarrow Mon \rightarrow Set). There is a functors $U: \mathbf{Grp} \rightarrow \mathbf{Mon}$ and $U': \mathbf{Mon} \rightarrow \mathbf{Set}$. Each group G is already a monoid, so just let $U(G) = G$, and every monoid has a carrier, so just let $U'(M) = M$. Each group homomorphism is already a monoid homomorphism, so just let $U(f) = f$, and every monoid homomorphism includes a function between carrier sets, so just let $U'(f) = f$. These functors preserve identities and composition.

In fact, all U does is take a group and forget that it's a group, just remembering that it's a monoid. And all U' does is take a monoid and forget that it's a monoid, just remembering that it's got a set. So U and U' are called forgetful functors. I used the letter U because we call $U(G)$ the *underlying monoid* of the group G , and we call $U'(M)$ the *underlying set* of the monoid M .

Definition 2.11 (Representable functor). Let \mathcal{C} be a category, and let $c \in \mathcal{C}$ be an object. The *representable functor on c* is the functor $y_c: \mathcal{C} \rightarrow \mathbf{Set}$ given as follows:

1. The on-objects part $\text{Ob}(\mathcal{C}) \rightarrow \text{Ob}(\mathbf{Set})$ is given by

$$y_c(d) := \text{Mor}_{\mathcal{C}}(c, d)$$

2. The on-morphisms part $\text{Mor}_C(d_1, d_2) \rightarrow \text{Mor}_{\text{Set}}(y_c(d_1), y_c(d_2))$ sends an element $f: d_1 \rightarrow d_2$ to the function $\text{Mor}_C(c, d_1) \rightarrow \text{Mor}_C(c, d_2)$, which in turn sends an morphism $g: c \rightarrow d_1$ to the morphism $g * f: c \rightarrow d_2$.

One can check that $y_c(\text{id}_d) = \text{id}_{y_c(d)}$ and $y_c(f_1 * f_2) = y_c(f_1) * y_c(f_2)$, so it really is a functor.

Example 2.12. Let $(M, \text{id}, *)$ be a monoid, and consider the associated category \mathcal{M} with one object; see Example 2.4. It has only one object, namely \cdot , so what is the representable functor on \cdot ? It is a functor $y: \mathcal{M} \rightarrow \mathbf{Set}$ given by $y(\cdot) = \text{Mor}_{\mathcal{M}}(\cdot, \cdot) = M$. For any element $m \in M$, we have a function $y(m): M \rightarrow M$, given by $y(m)(m') = m' * m$.

Example 2.13. Let $(G, \text{id}, *)$ be a group, and consider the associated category \mathcal{G} with one object. A functor $A: \mathcal{G} \rightarrow \mathbf{Grp}$ sends the unique object of \mathcal{G} to a certain group; let's call it H . It also sends every morphism in \mathcal{G} , i.e. every element of G , to a group homomorphism $H \rightarrow H$. But since each element $g \in G$ is invertible, and since functors preserve composition and identity, we have $\text{id}_H = A(\text{id}) = A(g * g^{-1}) = A(g) * A(g^{-1})$, so $A(g)$ is an automorphism of H . Thus the functor A is a group homomorphism $G \rightarrow \text{Aut}(H)$, in disguise.

3 The Grothendieck construction and the Cayley graph

Definition 3.1 (Grothendieck construction). Let C be a category and let $F: C \rightarrow \mathbf{Set}$ be a functor. The *Grothendieck construction* of F is a category $\int F$, defined as follows:

1. $\text{Ob}(\int F) := \{(c, x) \mid c \in \text{Ob}(C), x \in F(c)\}$
2. $\text{Mor}_{\int F}((c, x), (c', x')) := \{g: c \rightarrow c' \mid F(g)(x) = x'\}$,

The identity and composition are taken from C .

Remark 3.2. If C has a set of generating morphisms A , then so does $\int F$. Namely, a morphism $f: (c, x) \rightarrow (c', x')$ is a generator iff $f: c \rightarrow c'$ is in A .

Proposition 3.3. *Let M be a monoid (or a group), and $A \subseteq M$ a set of generators. Consider the associated category \mathcal{M} with one object; see Example 2.4, and let $y: \mathcal{M} \rightarrow \mathbf{Set}$ be the unique representable functor. Then the Cayley graph for M with generators A is the underlying graph of the category-with-generators $\int y$.*