A higher-order temporal logic for dynamical systems

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Outline

Temporal Type Theory, `https://arxiv.org/abs/1710.10258`

An example system

The National Airspace System (NAS).

- Goals of NextGen:
  - Double the number of airplanes in the sky;
  - Remain extremely safe.

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- Systems of systems:
  - A great variety of interconnected systems.
  - Work in concert to enforce global property: safe separation.

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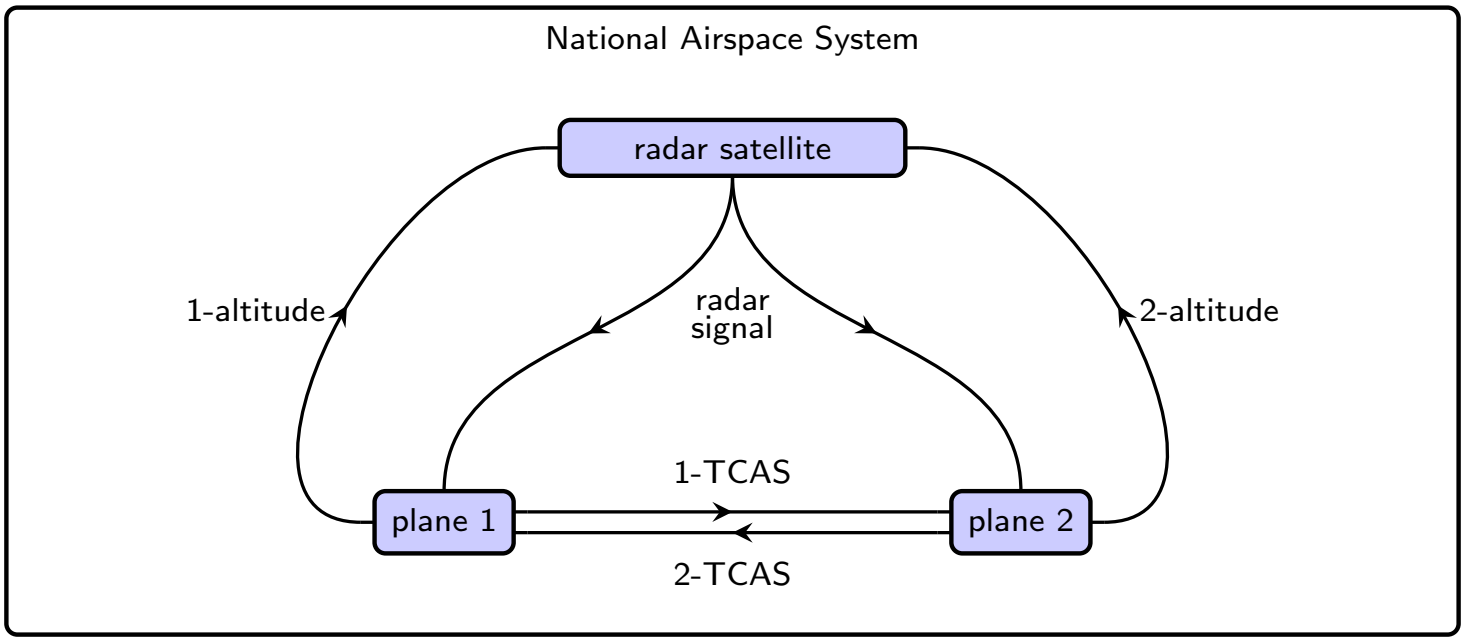
Systems of interacting systems in the NAS



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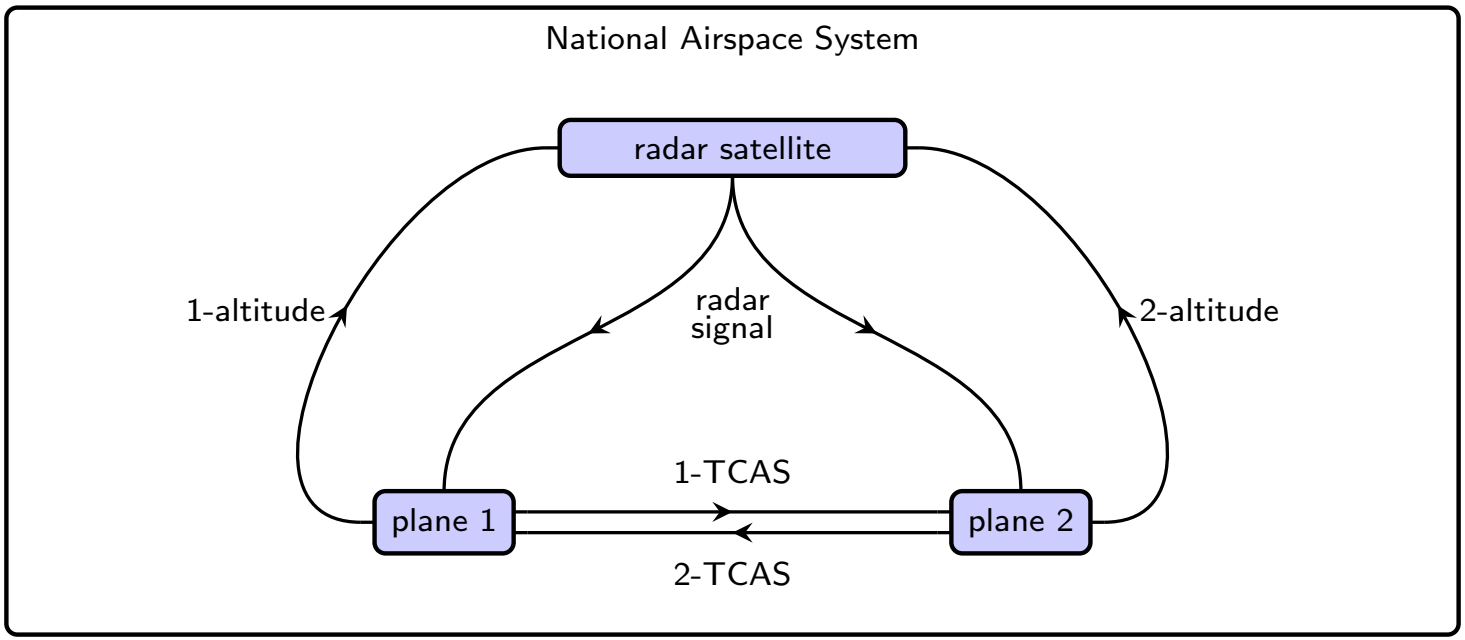
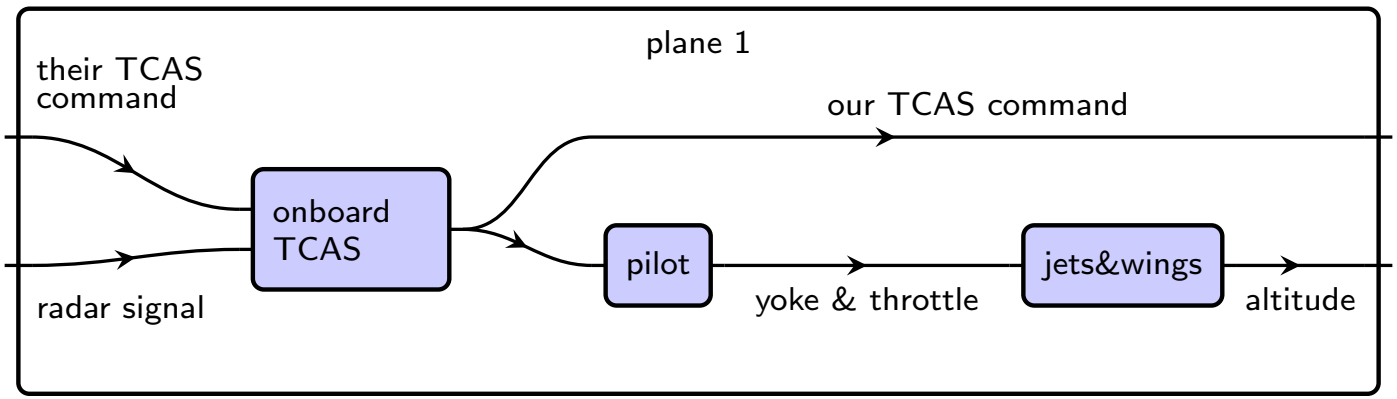
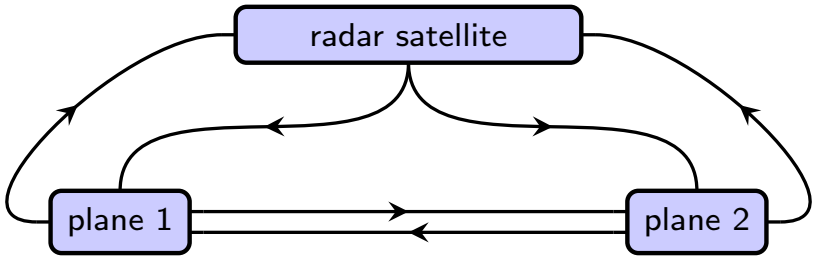




A hypergraph category



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  - Wires with arrows indicate "signal passing".

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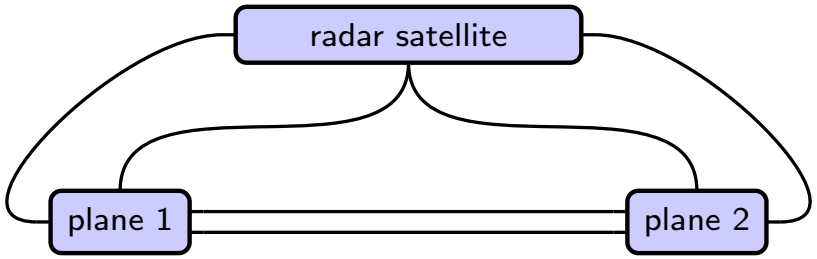

- What are these pictures?
  - Wires with arrows indicate "signal passing".
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  - Either way, the planes and the radars are *constraints*.
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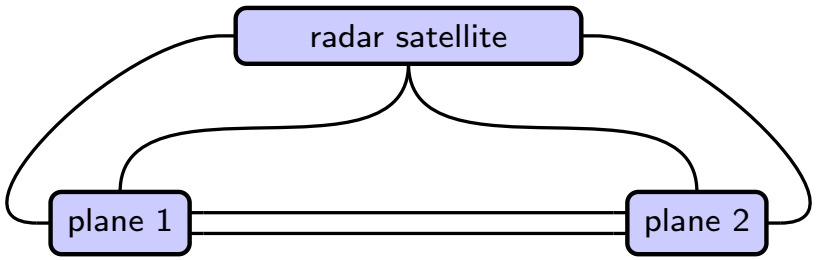
- What are these pictures formally?
  - Composition diagrams in a *hypergraph category*.
  - Example of a hypergraph category: relations.
    - where each wire is a set,
    - and each box is a relation $R \subseteq A_1 \times \cdots \times A_n$.

Relations in a topos



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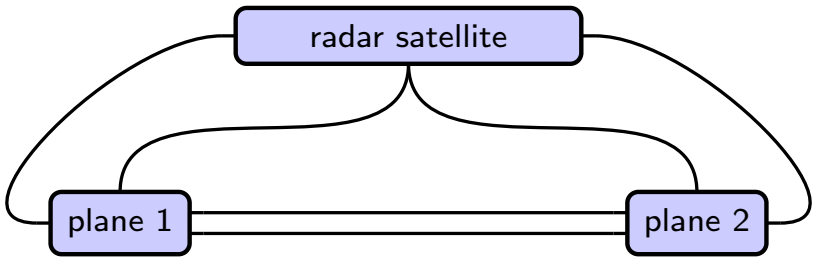

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- We talked about relations in $\mathcal{E} = \mathbf{Set}$.
- Idea generalizes to arbitrary toposes.
- Every topos $\mathcal{E}$ has a subobject classifier Ω
- Relations on $A = A_1 \times \cdots \times A_n$ are morphisms $A \longrightarrow \Omega$.

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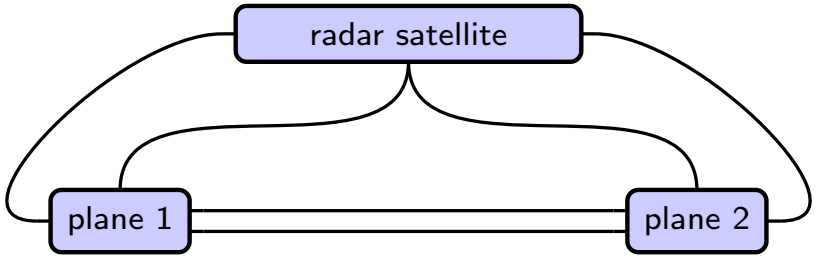

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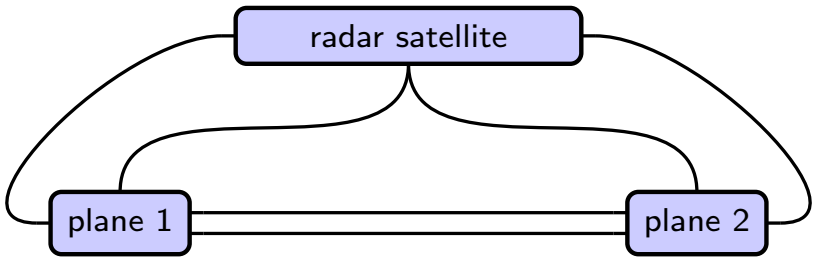

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- They live in time.
- Goal: a good topos for studying behaviors (hence time).

NAS use-case as guide

What's the topos for the National Airspace System?

- This question was a major guide for our work.
- Need to combine many common frameworks into a "big tent".
  - Differential equations, continuous dynamical systems.
  - Labeled transition systems, discrete dynamical systems.
  - Delays, non-instantaneous rules.

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Relationship to toposes:

- Toposes have an associated internal language and logic.
- Can use formal methods (proof assistants) to prove properties of NAS.

Plan of the talk

1. Define a topos $\mathcal{B}$ of behavior types.

2. Discuss *temporal type theory*, which is sound in $\mathcal{B}$.

3. Return to our NAS use-case.

Outline

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What is a topos and why?

Toposes—invented by Grothendieck—generalize topological spaces.

- Basic idea:
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- Two examples: topological spaces and databases.
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Question: What's a good site at which *behaviors* live?

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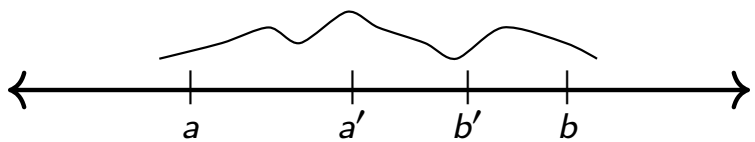
  - Compatibility:
    - Restriction maps compose: $B(a, b) \longrightarrow B(a', b') \longrightarrow B(a'', b'')$.
  - Gluing conditions:
    - "Continuity": $B(a, b) = \lim_{a<a'<b'<b} B(a', b')$.
    - "Composition": $B(a, b) = B(a, b') \times_{B(a', b')} B(a', b)$.



Why $\mathbb{R}$ is not preferable as the site

Two reasons *not to use* $\mathrm{Shv}(\mathbb{R})$ as our topos.

- Want behavior types with non-composable behaviors
  - "Roughly monotonic": $\forall(t_1, t_2).\, t_1 + 5 \leq t_2 \Rightarrow f(t_1) \leq f(t_2)$.
  - "Bounded distance": $\forall(t_1, t_2).\, -5 < f(t_1) - f(t_2) < 5$.
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We'll discard composition gluing and add translation invariance.

The interval domain $\mathbb{IR}$

Discarding composition gluing, we have the site:

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This corresponds to what's called the *interval domain* $\mathbb{IR}$.

- It's a topological space, not Hausdorff, but sober.
- Points: intervals $[d, u]$ (down/up).
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Specialization order on points: $[d, u] \sqsubseteq [d', u']$ means $d \leq d' \leq u' \leq u$.

$\mathbb{IR}$ and the upper half-plane

We'll visualize $\mathbb{IR}$ using the upper half-plane, $H = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y\}$.

Idea: midpoint and radius.... Let's replace variables.

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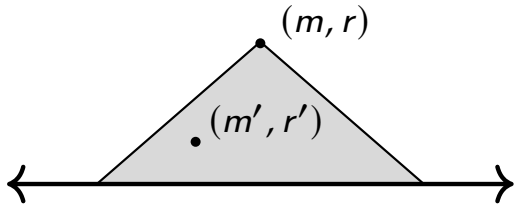
- There's a continuous bijection $H \xrightarrow{\cong} \mathbb{IR}$.
- Map forward: $(m, r) \mapsto [m + r, m - r]$.
- Map backward: $[d, u] \mapsto (\frac{u+d}{2}, \frac{u-d}{2})$.

There's a bijection, but H has more open sets.

Order and opens in $\mathbb{IR}$, from the half-plane perspective

$(m, r) \sqsubseteq (m', r')$ is this sort of "cone" relationship



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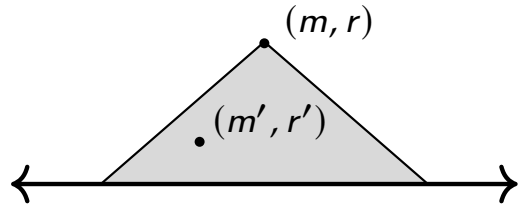

The opens are in bijection with Lipschitz functions $f : \mathbb{R} \to \mathbb{R}^+$.

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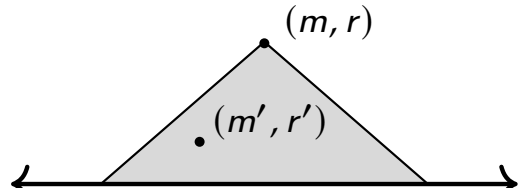

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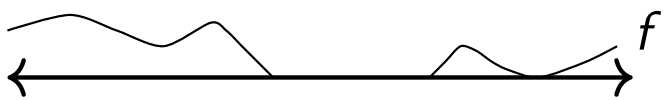
Here's a picture of an open set:



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  - Call $\mathbb{IR}_{/\triangleright}$ the site of *translation-invariant intervals*.
- We'll see that the associated topos has an object `Time`.
- We can use it to look at behaviors over different time-windows.

Our choice of topos $\mathcal{B}$

Equivalently, $\mathbb{IR}_{/\triangleright}$ is the following site:

- Objects $= \{\ell \in \mathbb{R}_{\geq 0}\}$.
- $\mathrm{Hom}(\ell', \ell) = \{\langle r, s\rangle \mid r + \ell' + s = \ell\}$

- Coverage $\{\langle r, s\rangle : \ell' \longrightarrow \ell \mid r > 0, s > 0\}$.
- When $r, s > 0$, write $\ell' \rightsquigarrow \ell$.[2]

The topos of behavior types: $\mathcal{B} = \mathrm{Shv}(\mathbb{IR}_{/\triangleright})$.

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Example $\mathtt{Time} \in \mathcal{B}$, the behavior type of a clock:

- $\mathtt{Time}(\ell) := \{(d, u) \in \mathbb{R}^2 \mid u - d = \ell\}$.
- For $\langle r, s\rangle : \ell' \to \ell$ the restriction is $(d, u)\big|_{\langle r,s\rangle} := (d + r, u - s)$.
- The required map is iso.

[2] Johnstone-Joyal's notation in "Continuous categories and exponentiable toposes".

Outline

Temporal Type Theory, `https://arxiv.org/abs/1710.10258`

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    - Types: $\mathbb{N}$, `Prop`, products, arrows, sums, quotients.
    - Terms: tupling, projection, lambda abstraction, evaluation, etc.
    - Propositions: $\exists, \forall, \wedge, \vee, \neg, \Rightarrow, \top, \bot$.
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I thought this was dreadfully boring. Until I witnessed...

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- Kripke-Joyal is a machine that turns logic into topos-proofs.
- Suppose you have any expression in the type theory.
  - It automatically has semantics in your topos.
  - That is, it means something about sheaves X.
  - $\forall(x : X)$ – "for all restriction maps and sections x..."
  - $\exists(x : X)$ – "there is a covering family and a section x in each..."
  - Each connective $\wedge, \vee, \Rightarrow$, means something sheafy.
- Statements and proofs are recursive, tree-like structures.
  - Kripke-Joyal recurses over that structure.
  - At each step, it unwinds the logic into restrictions, covers, sections.
  - It manages all the topos stuff and lets you believe you're in **Set**.

The Kripke-Joyal semantics: doing the heavy lifting.

Dedekind numeric objects

One can define the real numbers in the type theory.

- Follow Dedekind: a real is two subsets of $\mathbb{Q}$.
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  - Bounded: $\exists(q : \mathbb{Q}).\, \delta q$ and $\exists(q : \mathbb{Q}).\, \upsilon q$,
  - δ-Rounded: $\forall(q : \mathbb{Q}).\, \delta q \Leftrightarrow \exists(q' : \mathbb{Q}).\, (q < q') \wedge \delta q'$,
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- The sheaf $\mathbb{R}$ has a remarkably beautiful semantics.
  - Use Kripke-Joyal to unwind what this is. Find:
  - If $\mathcal{E} = \mathrm{Shv}(T)$ for some topological space T, ...
  - ... $\mathbb{R}$ = sheaf of continuous real-valued functions on T.
- Drop various axioms to get other interesting numeric objects.
  - Drop "bounded" and you get unbounded reals.
  - Drop "located" and you get intervals $\mathbb{IR}$, internally.

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One atomic type: $\mathtt{Time} \subseteq \mathbb{IR}$.

- Example axiom: $\mathtt{Time}$ is an $\mathbb{R}$-torsor:
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- With it, we can make temporal logic statements
  - $\forall(t_1, t_2 : \mathtt{Time}).\, (t_1 + 5 \leq t_2) \Rightarrow f(t_1) \leq f(t_2)$.
  - "The function f is roughly monotonic."
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Next up: using logic to define subtoposes.

Modalities in general

We refer to Lawvere-Tierney topologies as *modalities*.[3]

- $j : \texttt{Prop} \to \texttt{Prop}$ with $P \Rightarrow jP$, $jjP \Rightarrow jP$, and either/both of:

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  - $j(P \wedge Q) \Leftrightarrow (jP \wedge jQ)$.
  - $(P \Rightarrow Q) \Rightarrow (jP \Rightarrow jQ)$.
- The following are modalities for any $U : \texttt{Prop}$,
  - open: $o_U(P) := U \Rightarrow P$,
  - closed: $c_U(P) := U \vee P$,
  - quasi-closed: $q_U(P) := (P \Rightarrow U) \Rightarrow U$.
- Example: if $U = \bot$ then quasi-closed is double-negation.

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Modalities in our setting

$j : \texttt{Prop} \to \texttt{Prop}$ with $P \Rightarrow jP$, $jjP \Rightarrow jP$, $j(P \Rightarrow Q) \Rightarrow (jP \Rightarrow jQ)$.

- Example 1: "t is not not 0": $@_0 P := (P \Rightarrow t \# 0) \Rightarrow t \# 0$.
  - The topos of $@_0^t$ sheaves is an internal copy of **Set**.
  - $\mathbb{R}_{@0}$ is semantically the skypscraper sheaf $\mathbb{R}$ at 0.
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  - The topos of π-sheaves is an internal copy of $\mathrm{Shv}(\mathbb{R})$.
  - $\mathbb{R}_\pi$ is semantically the usual continuous real-valued functions.

The derivative

Internally define the derivative of $x : \mathbb{R}_\pi$.

- Approximate it with interval-valued functions y:
- Say y *approximates the derivative of* x if, for all $r_1 < r_2$ in $\mathbb{R}$,

$$y \sqsubseteq \frac{x^{\textcircled{o}}(r_2) - x^{\textcircled{o}}(r_1)}{r_2 - r_1}.$$

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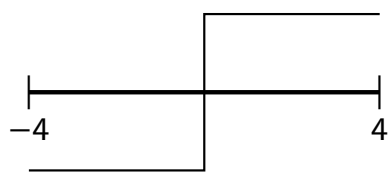
It looks familiar, and it works.

- Internally: linear, $\frac{d}{dt}(t) = 1$, Leibniz rule.
- Externally: it really is the derivative when derivative exists.

Even makes sense when derivative doesn't exist, e.g. absolute value:

$$\frac{d}{dt}|t| = \begin{cases} [-1,-1] & \text{if } t < 0 \\ [-1,1] & \text{if } t = 0 \\ [1,1] & \text{if } t > 0 \end{cases}$$



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As a logical expression, derivatives work like anything else.

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- Maybe a, b are continuous functions of time.
- Regardless, it's an equation in the logic.
  - Use it with $\top, \bot, \neg, \vee, \wedge, \Rightarrow, \exists, \forall$.
  - Can be combined with other properties.

Outline

Temporal Type Theory, `https://arxiv.org/abs/1710.10258`

The problem: safe altitude

Simplifying the safe separation problem.

- Real problem: safe separation for pairs of planes.
  - Components: Radars, pilots, thrusters/actuators.
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Goal: combine disparate guarantees to prove useful result.

Setup

Variables to be used, and their types:

$$t : \texttt{Time}. \quad T, P : \text{Cmnd}. \quad a : \mathbb{R}_\pi. \quad \texttt{safe}, \texttt{margin}, \texttt{del}, \texttt{rate} : \mathbb{Q}.$$

What these mean:

- $t : \texttt{Time}.$ time-line (a clock).
- $a : \mathbb{R}_\pi.$ altitude (continuously changing).
- $T : \text{Cmnd}.$ TCAS command (occurs at discrete instants).
- $P : \text{Cmnd}.$ pilot's command (occurs at discrete instants).
- $\texttt{safe} : \mathbb{Q}.$ safe altitude (constant).
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- Can prove safe separation

$$\forall(t : \texttt{Time}). \downarrow_0 (t > \texttt{del} + \frac{\texttt{safe}}{\texttt{rate}} \implies a \geq \texttt{safe}).$$

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Temporal Type Theory, `https://arxiv.org/abs/1710.10258`

Summary

- Idea: topos theory for integrating systems.
- Systems characterized by their possible behaviors.
  - $\mathcal{B} = \mathrm{Shv}(\mathbb{I}\mathbb{R}_{/\triangleright})$: topos of behavior types.
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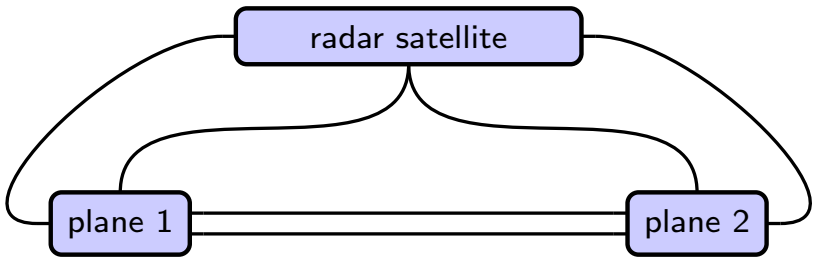

- Integrate systems in a hypergraph category of relations.
  - Each component X has an interface, say $A_1, \ldots, A_n$.
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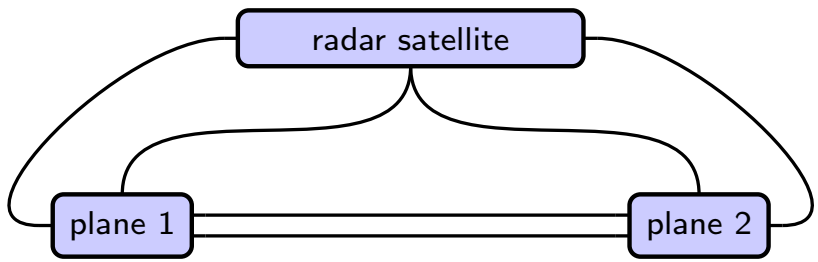

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- A new *temporal type theory* with sheaf semantics.

Questions and comments are welcome. Thanks!