Learning relations via adjunctions

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November 26, 2018

Abstract

Brendan and I have been developing a new categorical viewpoint—and graphical calculus for—regular logic. I've recently been thinking about how some of the surrounding ideas might apply to learning and adaptation via something I might call a "theory building" adjunction. Some of these ideas seem to connect to Reuben Cohn-Gordon's recent seminar talk on the "Gricean adjunction" from pragmatics.

The ideas I'll present are at an early stage in terms of maturity, and the goal of the talk is to be fun and pictorial. Still, I will certainly make connections to the underlying mathematical abstractions.

Learning relations via adjunctions

I. Introduction — 4 mins

   A. Learning and logic (Aristotle, rules for thinking)
   B. Adaptation: conforming internal relations to changing environment
   C. Logic: rules for combining relations.

II. Regular logic and a graphical calculus

   A. Regular logic with pictures — 12 mins
      1. Types Λ, contexts $\Gamma = (n \xrightarrow{v} \Lambda)$
      2. Poset $T(\Gamma)$ of *relations* in context Γ
         a. Order called "entailment",
         b. Denote by $\varphi \vdash \psi$.
      3. Rules for moving relations between contexts
         a. Combining contexts $T(\Gamma) \times T(\Gamma') \xrightarrow{\wedge} T(\Gamma \oplus \Gamma')$ and $1 \xrightarrow{\texttt{true}} T(0)$
         b. Add var., Equate vars, Split var., Quantify-out var.
      4. Rules for how these combine, e.g. $\exists x'.\, \varphi(x') \wedge (x = x') \vdash_x \varphi(x)$.
   B. Regular categories — 8 mins
      1. Category with: finite limits and pullback-stable image factorizations
      2. Examples:
         a. Set, FinSet, $\mathsf{Set}^{\mathrm{op}}$, Grp
         b. Any equational theory,
         c. Any "limit-epi sketch" (database theory: "EDs")
      3. Raison d'être: bicategory $\mathcal{R}\mathsf{el}_{\mathscr{R}}$ of relations in $\mathscr{R}$
         a. Same objects, relations as morphisms
         b. Composition via pullback and image factorization
         c. Morphisms in $\mathscr{R}$ are the left adjoints in $\mathcal{R}\mathsf{el}_{\mathscr{R}}$
   C. Regular logic is internal language of reg. cat — 3 mins
      1. All "rules for moving" are pullback, epi-mono, etc.
      2. A regular theory has a "syntactic category"
      3. Theory of subobjects in regular category is a regular theory

III. Graphical calculus and extension to geometric logic

   A. Graphical calculus (Λ, T): — 10 mins

1. Ajax functor $T\colon \mathcal{R}\mathsf{el}_{\mathsf{FRgC}(\Lambda)} \to \mathcal{P}\mathsf{oset}$
2. Here $\mathsf{FRgC}(\Lambda)$ is free regular category on Λ
   a. Objects in $\mathsf{FRgC}(\Lambda)$ are *contexts* Γ.
   b. Very similar to $\mathsf{FinSet}_{/\Lambda}$.
   c. Construction: $\mathsf{FRgC}(\Lambda) = (\mathsf{FinSet} \downarrow \mathbb{P}_f(\Lambda))^{\mathrm{op}}$
   d. Comma categories of regular cats are regular
   e. E.g. $n = 1$: $\begin{tikzcd} (0,\varnothing) & (0,\{1\}) \arrow[l, tail] & (1,\{1\}) \arrow[l, two heads] \arrow[r, tail] & (2,\{1\}) \arrow[l, two heads, shift right=2] \arrow[l, two heads, shift left=2] \end{tikzcd}$
3. Relations in $\mathsf{FRgC}(\Lambda)$
   a. Denote $\mathcal{R}\mathsf{el}_{\mathsf{FRgC}(\Lambda)}$
   b. Draw as wiring diagrams without extra dots
4. Ajax functor $T\colon \mathcal{R}\mathsf{el}_{\mathsf{FRgC}(\Lambda)} \to \mathcal{P}\mathsf{oset}$
   a. Assigns a poset $T(\Gamma)$ to each context Γ
   b. Regular logic rules encoded in the functoriality and ajax-ness
5. Theorem: adjunction $\begin{tikzcd} \mathsf{RgCalc} \arrow[r, shift left=2, "\mathbf{syn}"] \arrow[r, phantom, "\Rightarrow"] & \mathsf{RgCat} \arrow[l, shift left=2, "\mathbf{rels}"] \end{tikzcd}$ s.th. for any $\mathcal{R} \in \mathsf{RgCat}$, there's an equivalence $\mathbf{syn}(\mathbf{rels}(\mathcal{R})) \simeq \mathcal{R}$.

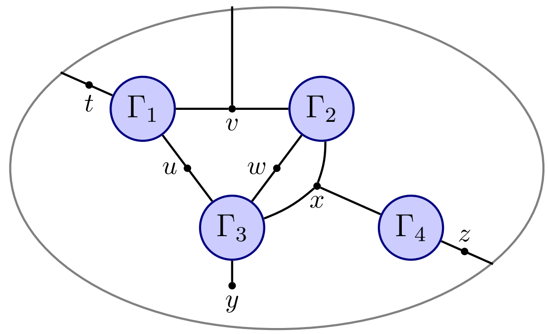
B. Extension to geometric logic — 8 mins

1. Coherent logic adds ORs, geometric adds infinitary ORs.
   a. Theory of groups is regular
   b. Theory of fields is coherent, $\texttt{true} \vdash_x (x = 0) \vee \exists y.\, x * y = 1$
   c. Theory of fields of finite characteristic is geometric, $\texttt{true} \vdash_x \bigvee_{n \geq 1} n \cdot x = 1$.
2. Sup-lattices
   a. Sup-lattices have all joins; morphisms preserve all joins
   b. Geometric categories: regular cats with sup-lattices as subobject posets
   c. Every topos is a geometric category

$$\begin{tikzcd} & \mathcal{S}\mathsf{upLat} \arrow[d, "U"] \\ \mathcal{R}\mathsf{el}_{\mathsf{FRgC}(\Lambda)} \arrow[ur, "T\ (\mathrm{lax})"] \arrow[r, "(\mathrm{ajax})"'] & \mathcal{P}\mathsf{oset} \end{tikzcd}$$

IV. Learning relations via adjunctions

$\varphi =$



$: \Gamma_1 \oplus \Gamma_2 \oplus \Gamma_3 \oplus \Gamma_4 \to \Gamma$

$$T(\varphi)\colon T(\Gamma_1) \otimes T(\Gamma_2) \otimes T(\Gamma_3) \otimes T(\Gamma_4) \to T(\Gamma)$$

A. This is a map of sup-lattices. 4 mins

1. Thus it preserves all joins.
2. Thus it is a left adjoint!
3. Let's write $T(\varphi)$ as φ^*. Then we have

$$T(\Gamma_1) \otimes \cdots \otimes T(\Gamma_4) \mathrel{\substack{\xrightarrow{\varphi^*} \\ \Rightarrow \\ \xleftarrow[\varphi_*]{}}} T(\Gamma)$$

B. So given relations $R_i \in T(\Gamma_i)$ and $E \in T(\Gamma)$, have 6 mins

$$\varphi^*(R_1, R_2, R_3, R_4) \vdash E \qquad \text{iff} \qquad (R_1, R_2, R_3, R_4) \vdash \varphi_*(E)$$

1. Elements in $T(\Gamma_1) \otimes \cdots \otimes T(\Gamma_4)$ are formal "ORs" of such (R_1, R_2, R_3, R_4)'s
2. To learn external relation E:
   a. try various combinations of relations R_i in each context Γ_i
   b. reject those that don't fit.
3. Gricean adjunction
   a. World validates the linguistic formula, $T(\varphi)(R) \vdash E$
   b. Internal relations are maximal: $T(\varphi)(R') \vdash E$ implies $R \vdash R'$.

C. Next step: Learn wiring pattern?

1. The wiring pattern is analogous to "neural architecture" from ML
2. But here it's a morphism in a regular cat $\mathsf{FRgC}(\Lambda)$
3. Can it be learned as well?