A higher-order temporal logic for dynamical systems

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An example system

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- Goals of NextGen:
    - Double the number of airplanes in the sky;
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- Systems of systems:
  - A great variety of interconnected systems.
  - Work in concert to enforce global property: safe separation.

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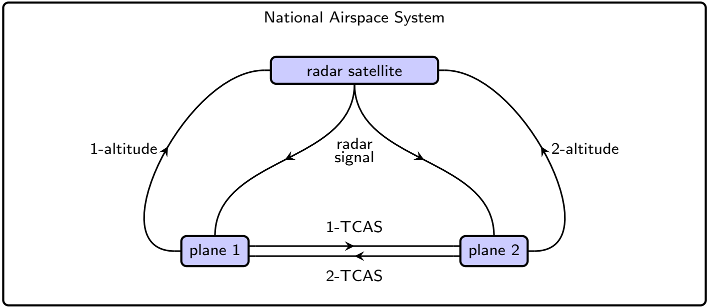
Systems of interacting systems in the NAS



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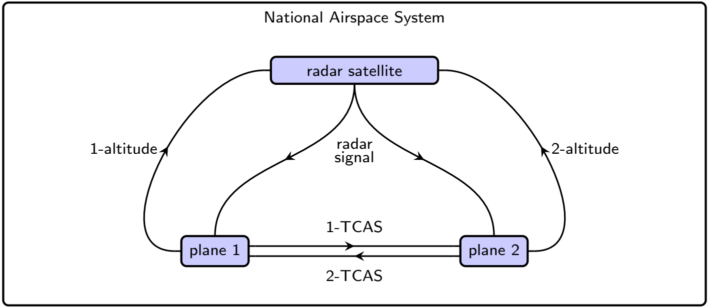

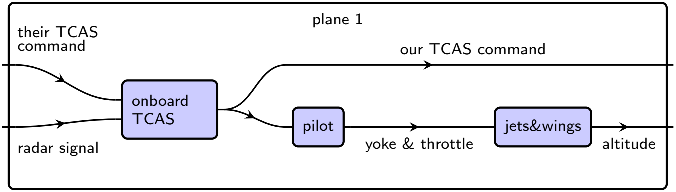

A poem

Anything that exists in time
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Its time-varying set
of behavioral possibilities
constitutes its type.

Behavior types

A *behavior type* T is a set of behavioral possibilities, changing in time.

- For each interval of time (a, b), a set $T(a, b)$.
- For each subinterval $(a', b') \subseteq (a, b)$, a function

$$\rho\colon T(a, b) \to T(a', b').$$

  "Restrict any T-possible movie clip to a sub-clip."
- And finally, $T(a, b) = \lim_{a<a'<b'<b} T(a', b')$.

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More general than hybrid: continuously overlapping, not piecewise, models.

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Sheaf / topos theory give a wealth of consequences for this commitment.

- Big one: a higher-order logic of time.
- A flexible system for building new behavior types from old.
- I'll discuss a new data structure for monitoring behavior.

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- Example: T is a bicycle, U is its pedals.
- A sheaf morphism $f\colon T \to U$ consists of:
  - ...for each interval (a, b), a function $f_{(a,b)}\colon T(a, b) \to U(a, b)$
  - ...such that all these 'commute' with restriction maps.

$$\begin{tikzcd} T(a,b) \arrow[r, "f_{(a,b)}"] \arrow[d, "\rho"'] & U(a,b) \arrow[d, "\rho"] \\ T(a',b') \arrow[r, "f_{(a',b')}"'] & U(a',b'). \end{tikzcd}$$

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- Variable sharing between X and Y is captured with sheaf morphisms

$$X \xrightarrow{f} V \xleftarrow{g} Y.$$

- Coupled system's behavior type is the sheaf $\{(x, y) \mid f(x) = g(y)\}$.

Mathematics in a topos

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- Sheaves act to an astounding degree like sets.
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All math constructions make sense in the topos of behavior types.

- Write usual def. of groups, rings, vector spaces, manifolds, etc.
- As sheaves, resulting mathematical objects *change through time*.
- I.e. groups, rings, etc. that change through time.

Logic for sets vs. sheaves

Like in sets, one can apply logic to sheaves.

- We have statements like $\forall(y : Y).\, \exists(x : X).\, f(x) = y$.
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Truth-values for sets are different than truth values for behaviors.

- The truth values for sets are: `true`, `false`.
- These make sense for timeless things, like properties of integers.
- Time-varying behaviors need time-varying truth values.

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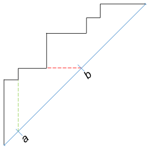
- Namely, plot at each moment “duration of perfect compliance.”



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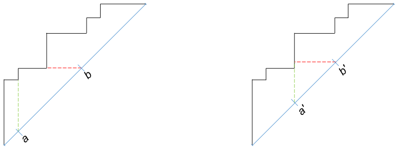

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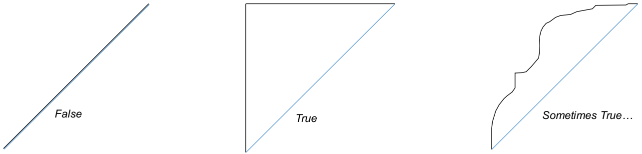
What are the truth values?

- Rather than `true` or `false` as truth values, ...
- ...the 'intervalic monitors' are the truth values of the topos.



Logic of behavior types

Logic involves operations and quantification.

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- Take $P \wedge Q$ to be $\min(P, Q)$.
- Take $P \vee Q$ to be $\max(P, Q)$.
- Similarly we can define the other connectives and quantification.
- All rules of standard higher-order logic hold for intervallic monitors.

HOTL: ordinary higher order logic + $\texttt{Time}$ + axioms

Higher-order logic is more expressive than first-order logic

- With HOL, you can reason over functions or subsets.
- Reasoning is almost the same, except constructive.
- This means you can use a proof assistant, like Coq or Lean.

$(\texttt{Threat} = 1) \Rightarrow \exists (p_1, p_2 : \mathsf{Pilot}).\, p_1 \neq p_2 \wedge \texttt{InFlight}(p_1) \wedge \texttt{InFlight}(p_2)$

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HOTL, our temporal Logic, adds a type $\mathtt{Time}$ and about 10 axioms.

- Behavior of $\mathtt{Time}$ is that of a perfect clock (with any $t = 0$ point).
- Example axiom: $\mathtt{Time}$ is an $\mathbb{R}$-torsor:

$$\forall (t : \mathtt{Time})(r : \mathbb{R}).\, \exists ! (t' : \mathtt{Time}).\, t + r = t'.$$

$$\forall (t, t' : \mathtt{Time}).\, \exists ! (r : \mathbb{R}).\, t + r = t'.$$

HOTL is quite expressive

Completely within the logic, one can define derivatives.

- This lets you define differential equations, inclusions, etc.
- Pure logic: $\forall(t : \mathtt{Time}).\, \exists(a : \mathbb{R}).\, 0 < \ddot{x} + \dot{x} - \sin(x^2) + a < 5t.$

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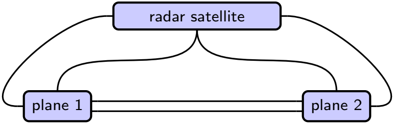
A “big tent” formalism for any sort of behavior.

Summary



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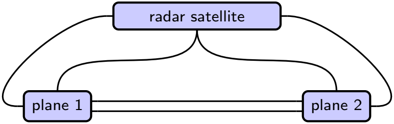

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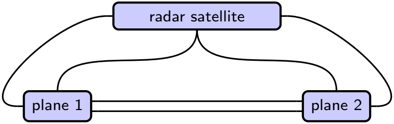

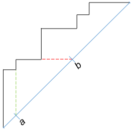

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  - Looks like usual set theory, $\forall, \exists, \wedge, \vee, \Rightarrow, \neg$; use formal methods
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  - Essential for capturing time-patterns: dwell-time, synchrony, etc.

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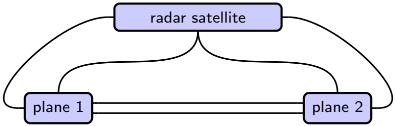

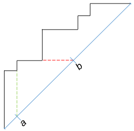

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- “Big tent”: import proofs from simpler systems and combine them.

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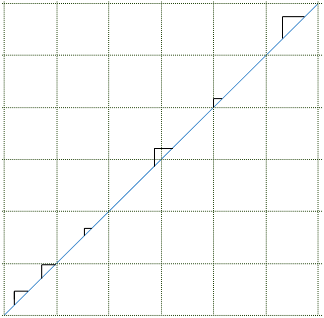
Questions and comments are welcome. Thanks!

Extra slide on intervallic monitors

Sample assertion: "I will visit you at least once per month."

$$\forall(t : \texttt{Time}).\, {\downarrow}^{t}_{[0,1]} \exists(r : \mathbb{R}).\, (0 < r < 1) \wedge @^{t}_{[r,r]}\texttt{Visit}$$

I'm visiting



I'm compliant with my assertion