A topos-theoretic approach to systems and behavior

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Outline

*Temporal Type Theory*, https://arxiv.org/abs/1710.10258

An example system

The National Airspace System (NAS)
- Goals of NextGen:
    - Double the number of airplanes in the sky;
    - Remain extremely safe.
- Safe separation problem:
    - Planes need to remain at a safe distance.
    - Can't generally communicate directly.
    - Use radars, pilots, ground control, radios, and TCAS.[1]
- Systems of systems:
    - A great variety of interconnected systems.
    - Work in concert to enforce global property: safe separation.

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[1]Traffic Collision Avoidance System.

Systems of interacting systems in the NAS
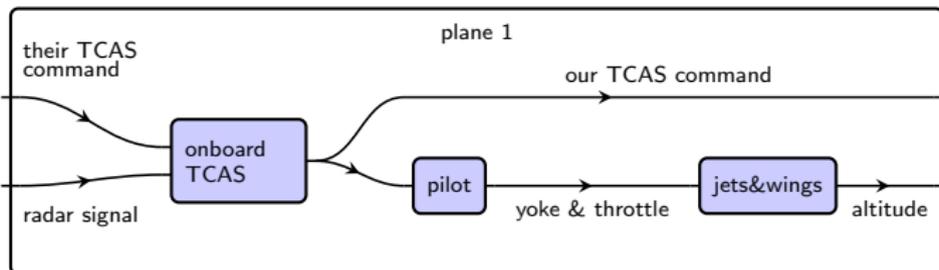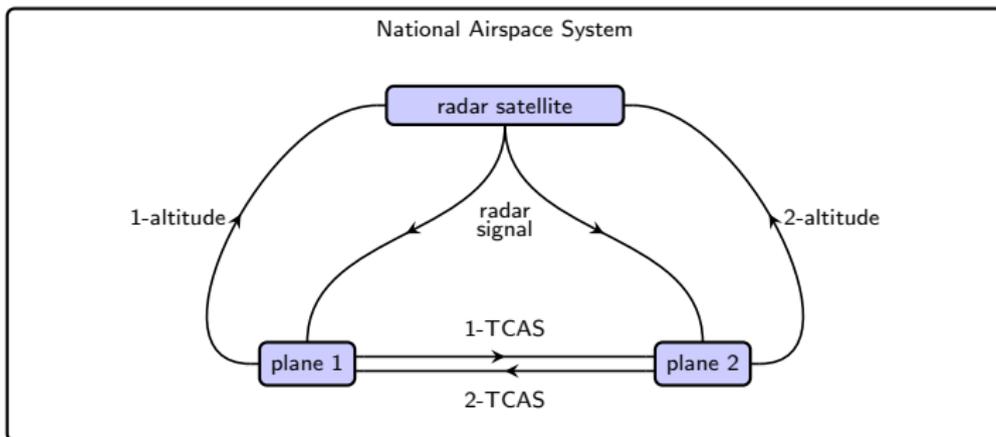
Behaviors as sheaves, "contracts" as predicates
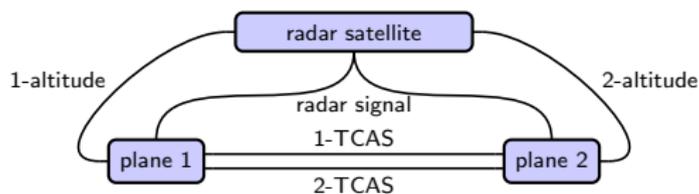
Everything in sight will be assigned a sheaf.

- A sheaf of possible behaviors for each box.
- A sheaf of possible behaviors (signals) for each wire.
- Sheaf morphisms from boxes to their wires.

A plane behavior has an associated altitude behavior, TCAS behavior, etc.

- Want to write it all logically and prove global property.
- Ask boxes to satisfy predicates="contracts"=relations on their wires.
- If everyone satisfies their contract, system maintains safe separation.

NAS use-case as guide

What's the topos for the National Airspace System?

- This question was a major guide for our work.
- Need to combine many common frameworks into a "big tent".
    - Differential equations, continuous dynamical systems.
    - Labeled transition systems, discrete dynamical systems.
    - Delays, non-instantaneous rules.
    - Determinism, non-determinism.
- Need a logic in which to prove safety of the combined system.
    - Currently, combination process takes place in engineers' heads.
    - For NextGen, we may need to do better.

Relationship to toposes:

- Toposes have an associated internal language and logic.
- Can use formal methods (proof assistants) to prove properties of NAS.

Plan of the talk

1. Define a topos $\mathcal{B}$ of behavior types.

2. Discuss *temporal type theory*, which is sound in $\mathcal{B}$.

3. Return to our NAS use-case.

Outline

1 **Introduction**

2 **The topos $\mathcal{B}$ of behavior types**
- Choosing a topos
- An intervallic time-line, $\mathbb{IR}$
- $\mathcal{B}$ the topos of behavior types

3 **Temporal type theory**

4 **Application to the NAS**

5 **Conclusion**

Temporal Type Theory, https://arxiv.org/abs/1710.10258

What is behavior?

We want to model behavior.

- What behaves in this sense?
    - You, your thoughts, your body, your airplane.
    - The radio, each movie, each fight, each fighter.
    - Any sort of thing that can "happen".

- What is a behavior type?
    - A behavior type is like "airplane behavior" or "pilot behavior"
    - Both are collections of possibilities, indexed by time intervals.
   - I want to conceptualize them as sheaves on time intervals.

So what should we mean by time?

- Only rule: whatever we mean, we should be able to capture:
    - Differential equations, labeled transition systems, delay...
    - ...compositionally: prove properties of combined systems.

First guess: $\mathbb{R}$ as timeline

$\mathbb{R}$ as timeline: Does it serve as a good site for behaviors?

- What would a behavior type $B \in \mathrm{Shv}(\mathbb{R})$ be?
    - On objects:
        - For each open interval $(a, b) \subseteq \mathbb{R}$, a set $B(a, b)$.
        - "The set of B-behaviors that can occur on (a, b)."
    - On morphisms:
        - For each $a \le a' < b' \le b$, a function $B(a, b) \to B(a', b')$
        - "The B-way to restrict ('clip') B-behaviors over subintervals."
    - Gluing conditions:
        - "Continuity": $B(a, b) = \lim_{a < a' < b' < b} B(a', b')$.
        - "Composition": $B(a, b) = B(a, b') \times_{B(a', b')} B(a', b)$.
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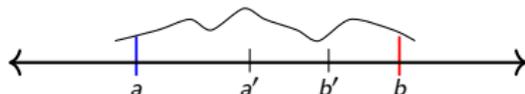
Why $\mathbb{R}$ is not preferable as the site

Two reasons *not to use* $\mathrm{Shv}(\mathbb{R})$ as our topos.

- 1. Often want to consider **non-composable** behaviors!
    - "Roughly monotonic": $\forall(t_1, t_2).\ t_1 + 5 \le t_2 \Rightarrow f(t_1) \le f(t_2)$.
    - "Don't move much": $\forall(t_1, t_2).\ -5 < f(t_1) - f(t_2) < 5$.
    - Neither of these have the "composition gluing".
- 2. Want to compare behavior across different time windows.
    - Example: a delay is "the same behavior at different times."
    - $\mathrm{Shv}(\mathbb{R})$ sees no relationship between $B(0, 3)$ and $B(2, 5)$.
    - Want "Translation invariance."

Solution:

- Replace $\mathbb{R}$ with an intervallic timeline.
- Quotient by translation action.

An intervallic time-line, $\mathbb{IR}$

For our timeline we use $\mathbb{IR}$ "the domain of real intervals".

- Definition $\mathbb{IR} = \mathrm{tw}(\mathbb{R}, \leq)^{\mathrm{op}}$.
    - Points: $\{[a, b] \mid a \leq b \in \mathbb{R}\}$.
    - $[a, b] \sqsubseteq [a', b']$ iff $a \leq a' \leq b' \leq b$.
    - $[a, b]$ is *less precise* than $[a', b']$.
    - $\mathbb{R} \subseteq \mathbb{IR}$ embeds as the maximal points, $[r, r]$.

- $\mathbb{IR}$ is a Scott domain:
    - Its poset of points determines a topology. How?
    - There's are adjunctions $\mathbb{IR} \;\begin{array}{c}\downarrow\\[-3pt]\xleftarrow{\text{colim}}\\[-6pt]\xrightarrow{\hspace{1.2cm}}\\[-3pt]\downarrow\end{array}\; \mathrm{Idl}(\mathbb{IR})$.
    - $[a, b] \in \Downarrow[a', b']$ iff $a < a' \leq b' < b$ (strict inequalities).
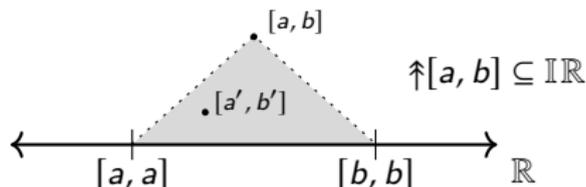   - Scott topology: take as basis of opens $\{\Uparrow[a, b] \mid a \leq b\}$.

This is our timeline: points are intervals.

Upper half-plane picture of $\mathbb{IR}$

Topologically, we can represent $\mathbb{IR}$ in the real upper half-plane.

- Here is $\Uparrow[a, b]$:

- Open sets $U \in \text{Op}(\mathbb{IR})$ are arbitrary unions of these.
- They have a nice characterization in terms of Lipschitz functions.
    - $\{U_f \in \text{Op}(\mathbb{IR})\} \cong \{f \colon \mathbb{R} \to \mathbb{R}_+ \mid f \text{ is 1-Lipschitz}\}$.
    - Points under curve f correspond to intervals (i.e. points) in U_f.

- These open sets will eventually be the truth-values in our topos.

Shv($\mathbb{IR}$): behaviors in the context of time

Each $X \in \text{Shv}(\mathbb{IR})$ is a behavior type occurring *in the context of time*.

- $\mathbb{IR}$ is our (intervallic) time-line.
- $X(a, b)$ is the set of X-behaviors over the interval (a, b).
- We can restrict behaviors to subintervals $a \le a' \le b' \le b$.

The truth-values in the topos $\text{Shv}(\mathbb{IR})$ are Scott-open sets.

- The area under a 1-Lipschitz function is a Scott open.
- Truth of any proposition (e.g. "roughly monotonic") is such an open.
    - Not "*is* its behavior roughly monotonic"?
    - But instead "*over what intervals* is it roughly monotonic"?

$\text{Shv}(\mathbb{IR})$ is the topos of behavior types in the context of time.

Next up: keep durations, remove fixed timeline.

Translation-invariant quotient topos $\mathcal{B}$

We want translation-invariance to compare behaviors over different times.

- Translation action $\mathbb{R} \overset{\triangleright}{\to} \mathrm{Aut}(\mathbb{IR})$, $r \triangleright (a, b) \coloneqq (a + r, b + r)$

- This induces a *left-exact comonad* T on $\mathrm{Shv}(\mathbb{IR})$.
    - (Left-exact comonads are what define geometric surjections.)
    - For $X \in \mathrm{Shv}(\mathbb{IR})$, define $TX \in \mathrm{Shv}(\mathbb{IR})$ by
    $$(TX)(a, b) \coloneqq \prod_{r \in \mathbb{R}} X(a + r, b + r).$$

- T-coalgebras are translation-equivariant sheaves.
- Define topos $\mathcal{B} \coloneqq T\text{-coAlg}$ of "behavior types".
- In fact $\mathcal{B}$ is an étendue.

    - There is an inhabited object, which we call $\mathtt{Time} \in \mathcal{B}$,
    - And an equivalence $\mathrm{Shv}(\mathbb{IR}) \cong \mathcal{B}/\mathtt{Time}$.
    - Makes precise "$\mathrm{Shv}(\mathbb{IR})$ is behavior types in the context of time."
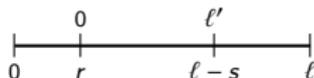
Next we'll give a site presentation of this topos $\mathcal{B}$.

A site for $\mathcal{B}$

Consider the twisted-arrow category $\mathbb{R}_{/\triangleright} = \mathrm{tw}(\mathbb{R}_{\geq 0})$.

- Objects $= \{\ell \in \mathbb{R}_{\geq 0}\}$.
- $\mathrm{Hom}(\ell', \ell) = \{\langle r, s \rangle \mid r + \ell' + s = \ell\}$.[2]

$\mathbb{R}_{/\triangleright}$ is a *continuous category* in the sense of Johnstone-Joyal.

- Coverage $\{\langle r, s \rangle \colon \ell' \to \ell \mid r > 0, s > 0\}$.
- When $r, s > 0$, write $\ell' \rightsquigarrow \ell$.

The topos of behavior types: $\mathcal{B} \cong \mathrm{Shv}(\mathbb{R}_{/\triangleright})$

- A sheaf X assigns a set of possible behaviors to each ℓ,
- And a restriction map to each included subinterval $\langle r, s \rangle \colon \ell' \to \ell$,
- Such that $X(\ell) \cong \lim_{\ell' \rightsquigarrow \ell} X(\ell')$.

Étendue means "extent"; $\mathbb{R}_{/\triangleright}$ is indeed extents (durations) of time.

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[2]Lawvere and others also studied sheaves on $\mathbb{R}_{/\triangleright}$, but used "composition gluing" whereas we use "continuity gluing."

Example behavior types $X \in \mathcal{B}$

We contend that any sort of behavior can be modeled as an object $X \in \mathcal{B}$.

- Trajectories through a vector field,
- Delays ($+$ delay differential equations),
- Stochastic walk through a graph,
- Ω: subobject classifier is "1-Lipschitz functions".
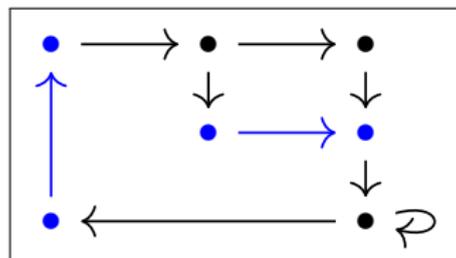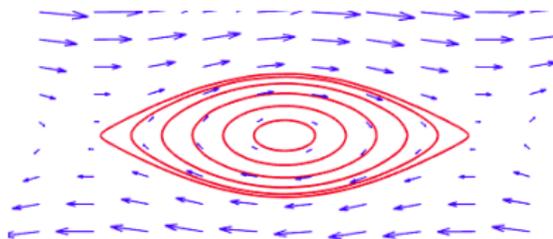
Next up: want logic to define other interesting behaviors.

Preview of higher-order temporal logic for behavior

Logical expressions give amazingly convenient representations.

- "Whenever I touch blue, I'll spend 1 full sec. on blue within 5 sec's."
- $\forall (t : \mathtt{Time}).\, @_{[0,0]}^{t} B(x) \Rightarrow \exists (r : \mathbb{R}).\, 0 \le r \le 5 \wedge @_{[r,r+1]}^{t} B(x).$

Kripke-Joyal semantics

- Logical expressions like the above can be interpreted in the topos $\mathcal{B}$.
- E.g. the above defines a map $P \colon X \to \Omega$, given $B \colon X \to \Omega$.
- This in turn gives a subtype $\{X \mid P\}$ of "P-good behavior".

How is internal logic is convenient?

- compact notation,
- precise semantics,
- quite expressive,
- readable in natural language, e.g. English.

Outline

1 **Introduction**

2 **The topos $\mathcal{B}$ of behavior types**

3 **Temporal type theory**
   - Toposes, type theory, and logic
   - A finitely-presented language with semantics in $\mathcal{B}$
   - Local reals and derivatives

4 **Application to the NAS**

5 **Conclusion**

Temporal Type Theory, https://arxiv.org/abs/1710.10258

Internal language of a topos

The internal language—previewed above—does a lot of heavy lifting.

- Here is *Kripke-Joyal semantics* for a sheaf topos.

Logical expr.	Sheaf-theoretic translation
$\forall(x : X).\, P(x)$	For all open U,[3] and all $x \in X(U)$, $P\big|_U(x)$ holds.
$\exists(x : X).\, P(x)$	There is an open cover $(U_i)_{i \in I}$ and a section $x \in X(U_i)$ in each, s.t. $P\big|_{U_i}(x_i)$ holds.
$P \Rightarrow Q$	For all open U, if $P\big|_U$ holds then $Q\big|_U$ holds.
$P \vee Q$	There is an open cover $(U_i)_{i \in I}$, s.t. $P\big|_{U_i}$ or $Q\big|_{U_i}$ for each i.
etc.	etc.

(In $\mathcal{B}$, all covers are filtered, so $\vee$ degenerates: no need for cover.)

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[3]Étendues are "locally locales", so we can use locale terminology, like "open subset".

Example: Dedekind numeric objects

In any sheaf topos, use logic to define various *Dedekind numeric objects*.

- Start with $\mathbb{Q}$; its semantics is the constant sheaf $\mathbb{Q}$.
- Consider functions $\delta : \mathbb{Q} \to \Omega$ ("the lower bounds" for some real).
- We can define the *lower reals* internally:

$$\underline{\mathbb{R}} := \{\delta : \mathbb{Q} \to \Omega \mid \exists q.\, \delta q \wedge \forall q.\, \delta q \Leftrightarrow \exists q'.\, q < q' \wedge \delta q'\}.$$

  - The semantics are nice on localic toposes. If X is a top. sp.,
  - $[\![\underline{\mathbb{R}}]\!](U) = \{\text{lower semi-continuous functions } U \to \mathbb{R}\}$.

- Dually, define $\bar{\mathbb{R}}$, with $[\![\bar{\mathbb{R}}]\!](U) = \{\text{upper semi-continuous}\dots\}$
- $\bar{\underline{\mathbb{R}}} := \underline{\mathbb{R}} \times \bar{\mathbb{R}}$: *extended intervals*.
- $\mathbb{R} := \{(\delta, \upsilon) : \bar{\underline{\mathbb{R}}} \mid \forall q.\, \neg(\delta q \wedge \upsilon q) \wedge \forall(q < q').\, \delta q \vee \upsilon q'\}$.

We refer to all of these as *Dedekind numeric objects*.

What is temporal type theory?

Temporal type theory: a finitely presented sublanguage of $\mathcal{B}$'s language.

- The internal language of $\mathcal{B}$ is infinite:
    - It consists of every object (as type), morphism (as term),
    - every commutative diagram, finite limit, exp'l object, etc. in $\mathcal{B}$.

- What if we want non-topos-theorists to use this formal system?
    - NASA uses *formal methods* to prove properties of systems.
    - These are formulas and proofs written in (temporal) logic.
    - We want same, but with richer type system, better semantics.

- We present a finite sub-language; build what we need from within.

This finite sublanguage is what we call *temporal type theory*.

Temporal type theory

The finitely presented language has:

- One atomic predicate symbol, $\mathrm{unit\_speed} \colon \bar{\bar{\mathbb{R}}} \to \Omega$.

  - From here, define $\mathtt{Time} := \{ x \colon \bar{\bar{\mathbb{R}}} \mid \mathrm{unit\_speed}(x) \}$.
  - Idea: internalize the set of time-lines (clock behaviors).
    - What is a clock behavior (on an external interval)?
    - It is an internal interval, moving along at unit speed.

- The theory has ten axioms, e.g. that $\mathtt{Time}$ is an $\mathbb{R}$-torsor:
  - $\forall (t \colon \mathtt{Time})(r \colon \mathbb{R}).\, t + r \in \mathtt{Time}$,
  - $\forall (t_1, t_2 \colon \mathtt{Time}).\, \exists !(r \colon \mathbb{R}).\, t_1 + r = t_2$.

Sound semantics in $\mathcal{B}$:

- We already had $\mathtt{Time} \in \mathcal{B}$ externally.
- Check that with that interpretation, the ten axioms hold.

Aside: relation to other temporal logics

There are other, widely used, temporal logics.

- They involve modalities like "Until" and "Since".
- Completeness results like Kamp's theorem:
    - Equivalence with "first-order monadic logic of order" $FO(<)$
        - Monadic doesn't mean monad,
        - It means there is one type: Time,
        - And every predicate symbol is unary $P(t)$ only.
        - Time is ordered: we have a relation $<$ on Time.
        - The logic is otherwise first-order and boolean.
    - Example: $\forall t.\, P(t) \Rightarrow \exists t'.\, t < t' \wedge Q(t')$.

TTT is pretty different:

- It's a type theory; we have many different types (sheaves).
- We have a higher order logic, with no "monadic"-type restrictions.
- We can embed $FO(<)$ into our language (just $\neg\neg$ everything).
- Trade-off: TTT is much more expressive; much less "automatable".

Modalities, @ and π

There are a number of useful modalities (Lawvere-Tierney topologies).

- Modalities are internal monads $j \colon \Omega \to \Omega$.
    - That is, $P \Rightarrow jP$, $jjP \Rightarrow jP$, $j(P \wedge Q) \Leftrightarrow (jP \wedge jQ)$.
    - One-to-one correspondence {modalities} $\cong$ {subtoposes}.

- Example 1,2: in the context of $t :$ Time, have $\downarrow_{[a,b]}^{t}, @_{[a,b]}^{t} \colon \Omega \to \Omega$.
    - $\downarrow_{[a,b]}^{t} P := P \vee (a < t \vee t < b)$.
    - $@_{[a,b]}^{t} P := (P \Rightarrow (a < t \vee t < b)) \Rightarrow (a < t \vee t < b)$.
    - These are hard to read, but correspond to useful subtoposes:
        - $@_{[a,b]}^{t}$ corresponds to single point subtopos $\{[a, b]\} \subseteq \mathbb{IR}$.
        - $\downarrow_{[a,b]}^{t}$ corresponds to its closure $\downarrow [a, b] \subseteq \mathbb{IR}$.

- Example 3: In empty context we have $\pi \colon \Omega \to \Omega$.
    - $\pi P := \forall (t :$ Time$). @_{[0,0]}^{t} P$.
    - Corresponds to the dense subtopos $\mathbb{R}_{/\triangleright} \subseteq \mathbb{IR}_{/\triangleright}$.

We can use these modalities to define *local Dedekind numeric types*.

Local Dedekind numeric types

For any j, we can define $\underline{\mathbb{R}}_j$, $\bar{\underline{\mathbb{R}}}_j$, $\bar{\bar{\underline{\mathbb{R}}}}_j$, $\mathbb{R}_j$, etc.

- j-logic: replace all connectives/quantifiers with their j-counterparts.
    - Each connective / quantifier satisfies a universal property,
    - Want same univ. property on j-closed propositions $P, Q \in \Omega_j$.
    - I.e. reflect logic of j-subtopos $\mathcal{B}_j$ into $\mathcal{B}$.
    - Example: define j-logic versions of Dedekind numeric types.

- $\underline{\mathbb{R}}_j := \{\delta \colon \mathbb{Q} \to \Omega_j \mid (j\exists q.\, \delta q) \wedge (\forall q.\, \delta q \Leftrightarrow j\exists q'.\, q < q' \wedge \delta q')\}$
    - When $j = \mathrm{id}$ this is lower semicontinuous fns on $\mathbb{IR}$.
    - When $j = \pi$, it's lower semicontinuous fns on $\mathbb{R}$.
    - When $j = \mathbb{Q}_{[a,b]}^t$, it's lower semicontinuous fns on a point.[4]

Now we are equipped to define derivatives.

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[4] $\mathbb{R}_{\mathbb{Q}[a,b]}$ is the skyscraper sheaf for the set $\mathbb{R}$ at the point $[a, b]$.

Derivatives of continuous reals

We can define derivatives internally.

- Semantics of $x : \mathbb{R}_\pi$ is continuous function of (pointwise) time.
    - Evaluation of x at a point $r : \mathbb{R}$ is given by $@_{[r,r]}x \in \mathbb{R}_{@[r,r]}$
    - We denote this $x^@(r)$.

- We define the derivative of any interval function $x : \bar{\bar{\mathbb{R}}}_\pi$.
    - Result is another interval function $\dot{x} : \bar{\bar{\mathbb{R}}}_\pi$, namely:
    - $q_1 < \dot{x} < q_2$ iff for all $r_1 < r_2 : \mathbb{R}$,

$$q_1 \ll \frac{x^@(r_2) - x^@(r_1)}{r_2 - r_1} \ll q_2.$$

- Theorem: $\dot{x}$ internally is linear in x and satisfies Leibniz rule.
- Theorem: $\dot{x}$ externally has semantics of derivative of x.
    - Caveat: $\dot{x}$ is defined for any cts x, even if non-differentiable.
    - When x is externally differentiable, $\dot{x}$ is its derivative.
    - When not, $\dot{x}$ is interval-valued "very reasonable" notion.

Differential equations

As a logical expression, derivatives work like anything else.

- Consider a differential equation, like

$$f(\dot{x}, \ddot{x}, a, b) = 0.$$

- Maybe $a, b : \mathbb{R}_\pi$ are continuous functions of time.
- Regardless, $f(\dot{x}, \ddot{x}, a, b) = 0$ is just a logical expression.
    - Similarly, $-2 < f(\dot{x}, \ddot{x}, a, b) < 2$ is just a logical expression.
    - Use it with $\top, \bot, \neg, \vee, \wedge, \Rightarrow, \exists, \forall$.
    - Can be combined with any other property.

Outline

Temporal Type Theory, https://arxiv.org/abs/1710.10258

The problem: safe altitude

Simplifying the safe separation problem.

- Real problem: safe separation for pairs of planes.
    - Components: Radars, pilots, thrusters/actuators.
    - Behavior types: Discrete signals, (continuous) diff-eqs, delays.
- Simplification: safe altitude for one plane.
    - One radar, one pilot, one thruster.
    - Same behavior types: discrete, continuous, delay.

Goal: combine disparate guarantees to prove useful result.

Setup

Variables to be used, and their types:

t : Time. T, P : Cmnd. $a : \mathbb{R}_\pi$.   safe, margin, del, rate : $\mathbb{Q}$.

What these mean:

- t : Time.   time-line   (a clock).
- $a : \mathbb{R}_\pi$.   altitude   (continuously changing).
- T : Cmnd.   TCAS command   (occurs at discrete instants).
- P : Cmnd.   pilot's command   (occurs at discrete instants).
- safe : $\mathbb{Q}$.   safe altitude   (constant).
- margin : $\mathbb{Q}$.   margin-of-error   (constant).
- del : $\mathbb{Q}$.   pilot delay   (constant).
- rate : $\mathbb{Q}$.   maximal ascent rate (constant).

Behavior contracts

- t : Time.   time-line   (a clock).
- $a : \mathbb{R}_\pi$.   altitude   (continuously changing).
- T : Cmnd.   TCAS command   (occurs at discrete instants).
- P : Cmnd.   pilot's command   (occurs at discrete instants).
- safe : $\mathbb{Q}$.   safe altitude   (constant).
- margin : $\mathbb{Q}$.   margin-of-error   (constant).
- del : $\mathbb{Q}$.   pilot delay   (constant).
- rate : $\mathbb{Q}$.   maximal ascent rate   (constant).

- $\theta_1 := (\text{margin} > 0) \wedge (a \geq 0)$.
- $\theta_2 := (a > \text{safe} + \text{margin} \Rightarrow T = \text{level})$.
- $\theta_2' := (a < \text{safe} + \text{margin} \Rightarrow T = \text{climb})$.
- $\theta_3 := (P = \text{level} \Rightarrow \dot{a} = 0) \wedge (P = \text{climb} \Rightarrow \dot{a} = \text{rate})$.
- $\theta_4 := \text{is\_delayed}(\text{del}, T, P)$.

θ_4 is an abbreviation for a longer logical condition.
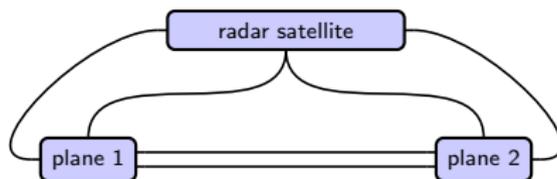
- Can prove safe separation
$$\forall (t : \text{Time}). \downarrow_0^t (t > \text{del} + \frac{\text{safe}}{\text{rate}} \ \Rightarrow \ a \geq \text{safe}).$$

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- Further reading

  Temporal Type Theory, https://arxiv.org/abs/1710.10258

Summary

- Idea: topos theory for integrating systems in a big tent framework.

- Many different formalisms for behavior, but they all occur in time.
    - We say that time occurs in intervals, which can be restricted.
    - Sheaves are behavior types: "what can occur over intervals."

- The topos has a native "internal" logic.
    - Looks like usual set theory, $\forall, \exists, \wedge, \vee, \Rightarrow, \neg$; use formal methods.
    - Has built-in Time object: do temporal logic.
    - Internal definition of ODEs, hybrid systems, etc.
    - Logically prove sheaf-theoretic behavioral properties.

This temporal type theory is quite general, and fully compositional.

If you're interested in reading more

- Book (published by Springer Berkhaüser).
    - *Temporal Type Theory*.
    - Freely available: https://arxiv.org/abs/1710.10258
    - Technical parts, some friendly parts.

- Book (to be published by Cambridge University Press).
    - *Seven Sketches in Compositionality*.
    - Freely available: https://arxiv.org/abs/1803.05316
    - Chapter 7 is about this material.
    - Friendly!

Questions and comments are welcome. Thanks!