A topos-theoretic approach to systems and behavior

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Toposes in Como
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Outline

Temporal Type Theory, `https://arxiv.org/abs/1710.10258`

An example system

The National Airspace System (NAS)

- Goals of NextGen:
  - Double the number of airplanes in the sky;
  - Remain extremely safe.

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- Systems of systems:
  - A great variety of interconnected systems.
  - Work in concert to enforce global property: safe separation.

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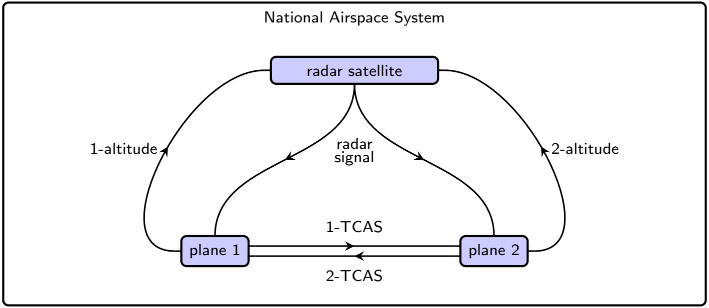
Systems of interacting systems in the NAS



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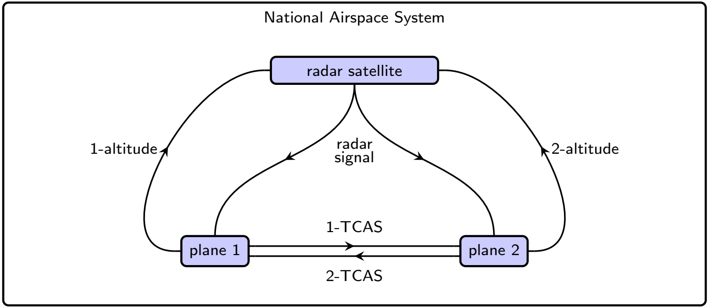
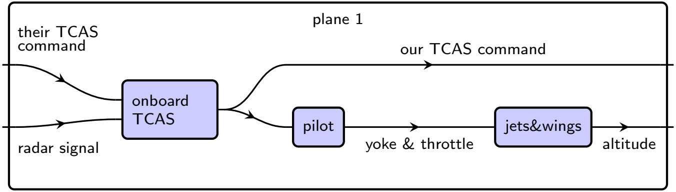
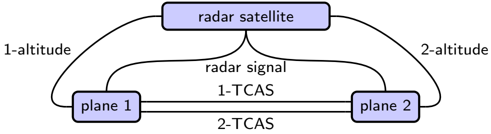




Behaviors as sheaves, “contracts” as predicates



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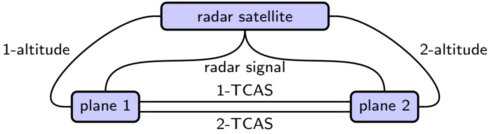

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A plane behavior has an associated altitude behavior, TCAS behavior, etc.

- Want to write it all logically and prove global property.
- Ask boxes to satisfy predicates=“contracts”=relations on their wires.
- If everyone satisfies their contract, system maintains safe separation.

NAS use-case as guide

What's the topos for the National Airspace System?

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  - Labeled transition systems, discrete dynamical systems.
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Relationship to toposes:

- Toposes have an associated internal language and logic.
- Can use formal methods (proof assistants) to prove properties of NAS.

Plan of the talk

1. Define a topos $\mathcal{B}$ of behavior types.
2. Discuss *temporal type theory*, which is sound in $\mathcal{B}$.
3. Return to our NAS use-case.

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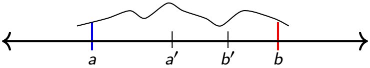

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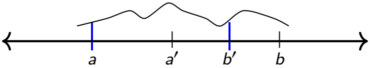

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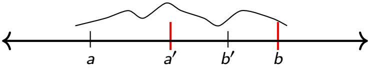

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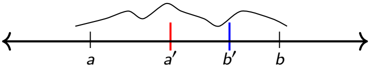

Why $\mathbb{R}$ is not preferable as the site

Two reasons *not to use* $\mathsf{Shv}(\mathbb{R})$ as our topos.

- 1. Often want to consider **non-composable** behaviors!
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Solution:

- Replace $\mathbb{R}$ with an intervallic timeline.
- Quotient by translation action.

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  - There's are adjunctions $\mathbb{IR} \underset{\twoheaddownarrow}{\overset{\downarrow}{\rightleftarrows}} \mathrm{Idl}(\mathbb{IR})$, with the leftward map labelled colim.
  - $[a, b] \in \twoheaddownarrow[a', b']$ iff $a < a' \leq b' < b$ (strict inequalities).
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This is our timeline: points are intervals.

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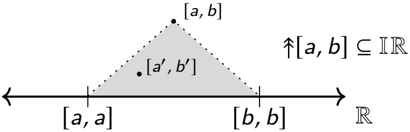

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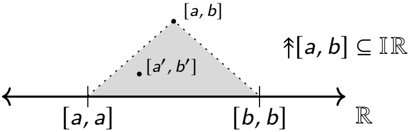

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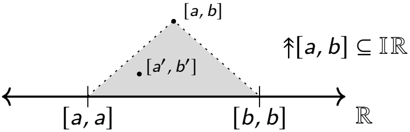

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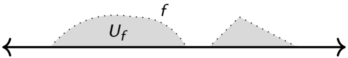
  - $\{U_f \in \mathrm{Op}(\mathbb{IR})\} \cong \{f : \mathbb{R} \to \mathbb{R}_+ \mid f \text{ is 1-Lipschitz}\}$.
  - Points under curve f correspond to intervals (i.e. points) in U_f.



- These open sets will eventually be the truth-values in our topos.

Shv($\mathbb{IR}$): behaviors in the context of time

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Next up: keep durations, remove fixed timeline.

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- This induces a *left-exact comonad* T on $\mathrm{Shv}(\mathbb{IR})$.
  - (Left-exact comonads are what define geometric surjections.)
  - For $X \in \mathrm{Shv}(\mathbb{IR})$, define $TX \in \mathrm{Shv}(\mathbb{IR})$ by

$$(TX)[a, b] := \prod_{r \in \mathbb{R}} X[a + r, b + r].$$

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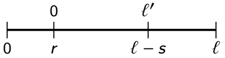
Next we'll give a site presentation of this topos $\mathcal{B}$.

A site for $\mathcal{B}$

Consider the twisted-arrow category $\mathbb{IR}_{/\rhd} = \mathrm{tw}(\mathbb{R}_{\geq 0})$.

- Objects $= \{\ell \in \mathbb{R}_{\geq 0}\}$.
- $\mathrm{Hom}(\ell', \ell) = \{\langle r, s\rangle \mid r + \ell' + s = \ell\}$[2]



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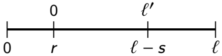

$\mathbb{IR}_{/\triangleright}$ is a *continuous category* in the sense of Johnstone-Joyal.

- Coverage $\{\langle r, s\rangle \colon \ell' \to \ell \mid r > 0, s > 0\}$.
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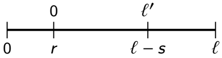

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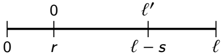

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Étendue means "extent"; $\mathbb{IR}_{/\triangleright}$ is indeed extents (durations) of time.

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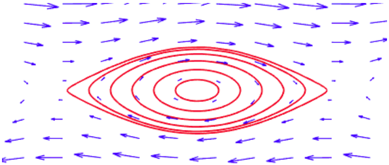

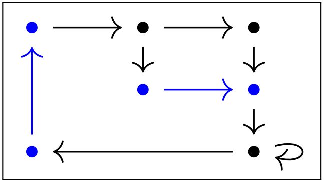

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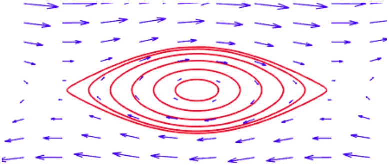

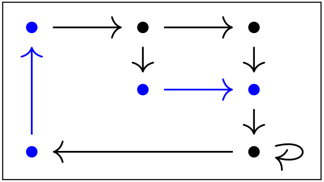

Next up: want logic to define other interesting behaviors.

Preview of higher-order temporal logic for behavior

Logical expressions give amazingly convenient representations.

- "Whenever I touch blue, I'll spend 1 full sec. on blue within 5 sec's."
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How is internal logic is convenient?

- compact notation,
- precise semantics,
- quite expressive,
- readable in natural language, e.g. English.

Outline

Temporal Type Theory, `https://arxiv.org/abs/1710.10258`

Internal language of a topos

The internal language—previewed above—does a lot of heavy lifting.

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$\exists(x:X).\,P(x)$	There is an open cover $(U_i)_{i\in I}$ and a section $x \in X(U_i)$ in each, s.t. $P\big|_{U_i}(x_i)$ holds.
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(In $\mathcal{B}$, all covers are filtered, so $\vee$ degenerates: no need for cover.)

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- Consider functions $\delta : \mathbb{Q} \to \Omega$ ("the lower bounds" for some real).
- We can define the *lower reals* internally:

$$\underline{\mathbb{R}} \coloneqq \{\delta : \mathbb{Q} \to \Omega \mid \exists q.\, \delta q \wedge \forall q.\, \delta q \Leftrightarrow \exists q'.\, q < q' \wedge \delta q'\}.$$

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- $\mathbb{R} := \{(\delta, \upsilon) : \overline{\underline{\mathbb{R}}} \mid \forall q.\, \neg(\delta q \wedge \upsilon q) \wedge \forall (q < q').\, \delta q \vee \upsilon q'\}$.

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We refer to all of these as *Dedekind numeric objects*.

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Temporal type theory: a finitely presented sublanguage of $\mathcal{B}$'s language.

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  - We want same, but with richer type system, better semantics.
- We present a finite sub-language; build what we need from within.

This finite sublanguage is what we call *temporal type theory*.

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  - From here, define `Time` $:= \{\, x : \underline{\bar{\mathbb{R}}} \mid \mathrm{unit\_speed}(x) \,\}$.
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Sound semantics in $\mathcal{B}$:

- We already had $\mathtt{Time} \in \mathcal{B}$ externally.
- Check that with that interpretation, the ten axioms hold.

Aside: relation to other temporal logics

There are other, widely used, temporal logics.

- They involve modalities like "Until" and "Since".
- Completeness results like Kamp's theorem:
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TTT is pretty different:

- It's a type theory; we have many different types (sheaves).
- We have a higher order logic, with no "monadic"-type restrictions.
- We can embed $FO(<)$ into our language (just $\neg\neg$ everything).
- Trade-off: TTT is much more expressive; much less "automatable".

Modalities, @ and π

There are a number of useful modalities (Lawvere-Tierney topologies).

- Modalities are internal monads $j\colon \Omega \to \Omega$.
  - That is, $P \Rightarrow jP, \quad jjP \Rightarrow jP, \quad j(P \wedge Q) \Leftrightarrow (jP \wedge jQ)$.
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- Example 3: In empty context we have $\pi\colon \Omega \to \Omega$.
  - $\pi P \coloneqq \forall(t : \mathtt{Time}).\, @^t_{[0,0]} P$.
  - Corresponds to the dense subtopos $\mathbb{R}_{/\rhd} \subseteq \mathbb{IR}_{/\rhd}$.

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- Example 3: In empty context we have $\pi\colon \Omega \to \Omega$.
  - $\pi P := \forall(t : \mathtt{Time}).\, @^t_{[0,0]} P$.
  - Corresponds to the dense subtopos $\mathbb{R}_{/\rhd} \subseteq \mathbb{IR}_{/\rhd}$.

We can use these modalities to define *local Dedekind numeric types*.

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  - When $j = \pi$, it's lower semicontinuous fns on $\mathbb{R}$.
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Now we are equipped to define derivatives.

Derivatives of continuous reals

We can define derivatives internally.

- Semantics of $x : \mathbb{R}_\pi$ is continuous function of (pointwise) time.
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  - When x is externally differentiable, $\dot{x}$ is its derivative.
  - When not, $\dot{x}$ is interval-valued "very reasonable" notion.

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- Maybe $a, b : \mathbb{R}_\pi$ are continuous functions of time.
- Regardless, $f(\dot{x}, \ddot{x}, a, b) = 0$ is just an equation in the logic.
  - Use it with $\top, \bot, \neg, \vee, \wedge, \Rightarrow, \exists, \forall$.
  - Can be combined with any other property.

Outline

Temporal Type Theory, `https://arxiv.org/abs/1710.10258`

The problem: safe altitude

Simplifying the safe separation problem.

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Goal: combine disparate guarantees to prove useful result.

Setup

Variables to be used, and their types:

$$t : \texttt{Time}. \quad T, P : \mathrm{Cmnd}. \quad a : \mathbb{R}_\pi. \quad \texttt{safe}, \texttt{margin}, \texttt{del}, \texttt{rate} : \mathbb{Q}.$$

What these mean:

■ $t : \texttt{Time}$.	time-line	(a clock).
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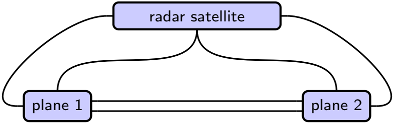
$$\forall (t : \mathtt{Time}). \downarrow_0^t \left(t > \mathtt{del} + \frac{\mathtt{safe}}{\mathtt{rate}} \;\Rightarrow\; a \geq \mathtt{safe}\right).$$

Outline

Temporal Type Theory, `https://arxiv.org/abs/1710.10258`

Summary



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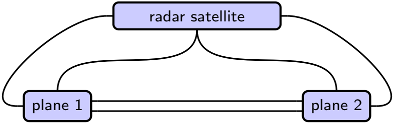

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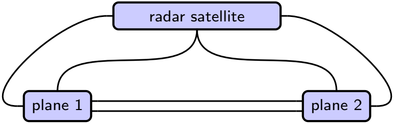

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- Many different formalisms for behavior, but they all occur in time.
  - We say that time occurs in intervals, which can be restricted.
  - Sheaves are behavior types: "what can occur over intervals."
- The topos has a native "internal" logic.
  - Looks like usual set theory, $\forall, \exists, \wedge, \vee, \Rightarrow, \neg$; use formal methods.
  - Has built-in `Time` object: do temporal logic.
  - Internal definition of ODEs, hybrid systems, etc.
  - Logically prove sheaf-theoretic behavioral properties.

This temporal type theory is quite general, and fully compositional.

If you're interested in reading more

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