Categorical views of regular, coherent, geometric logic: from classical to wiring-theoretic

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I. Introduction

A. Patterson's relational ologs [5]

1. Example: people who work in the same small company are acquainted [5]
2. Teaser: If David and Ryan are people who work at Ci, a small company, show they are acquainted [2]

B. Regular logic, coherent logic, geometric logic

1. Translate above into regular logic [2]
2. Regular formulas, regular sequents, proofs [2]
3. Examples of theories [4]
   a. groups and categories are regular
   b. fields and vector spaces are coherent
   c. torsion groups are geometric
   d. Today: focus on regular

C. Classical categorical formulation

1. Regular categories, coherent categories, geometric categories [1]
2. Definition of regular category: finite limits, image factorizations that are pullback stable. [3]
3. Key idea connecting to logic: morphisms can be characterized as relations with certain properties. [2]

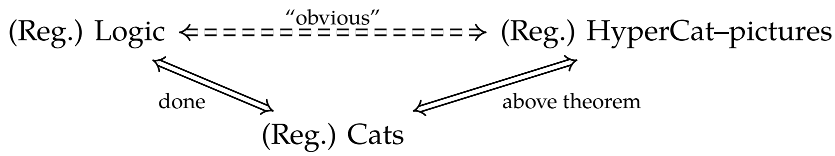
D. Theorem statement: [2]

1. Thm: Let $[A, B]$ denote the category of cocartesian operad functors $A \to B$. Then there is a monadic adjunction

$$\Phi\colon \int^{T\in\mathbf{Set}} [\mathbf{Cospan}_T, \wedge\text{-}\mathbf{SL}] \leftrightarrows \mathbf{RgCat} : \Psi$$

exhibiting **RgCat** as a reflective subcategory of **RgTh** $:= \int^{T\in\mathbf{Set}}[\mathbf{Cospan}_T, \wedge\text{-}\mathbf{SL}]$. In other words, Ψ is fully faithful.

2. We will unpack this.
3. Idea: [2]



II. The left-hand side

A. Unpacking

1. $\mathbf{Cospan}_T$ and wiring diagrams [3]
2. $\mathbf{Cospan}_T$ and ∧rm**SL** as cocartesian operads [3]
3. Change of types, $\int^{T\in\mathbf{Set}}\cdots$ [2]

B. Using the structure

1. Back to the teaser (use cocartesian structure) [4]
2. Exercise: prove the two definitions of "function" are equivalent. [2]

III. The maps Φ, Ψ

A. Ψ, fully faithful [4]

B. Φ [4]

IV. Conclusion

A. Generalizations [3]

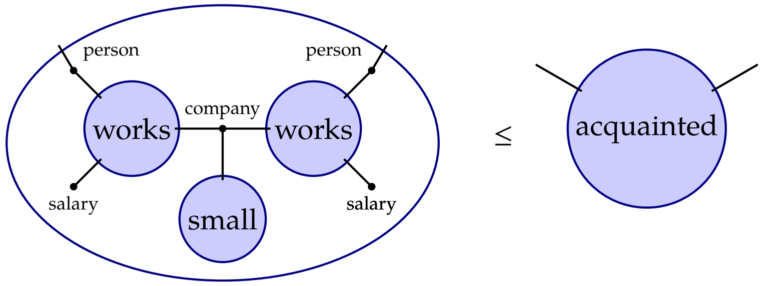
1. By correctly defining the operad of distributive lattices (really need operad structure to encode "multilinearity"), get

$$\Phi\colon \int^{T\in\mathbf{Set}}[\mathbf{Cospan}_T, \mathbf{Dist\text{-}Latt}] \leftrightarrows \mathbf{CohCat} : \Psi$$

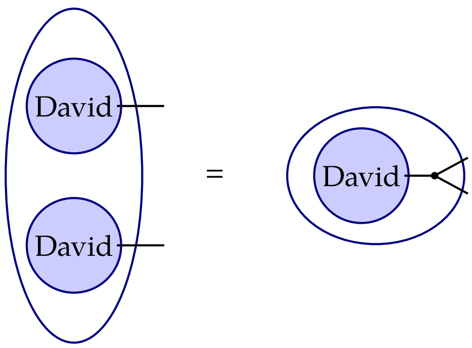
2. Similarly for geometric: use frames rather than distributive lattices
3. Operad of distributive lattices: $\phi\colon X_1, \ldots, X_n \to Y$ is a poset map $X_1 \times \cdots \times X_n \to Y$ that is join-preserving *in each variable*. Think "multilinear".

B. The point: graphical ways of thinking, still categorical, closer to the logic. [1]

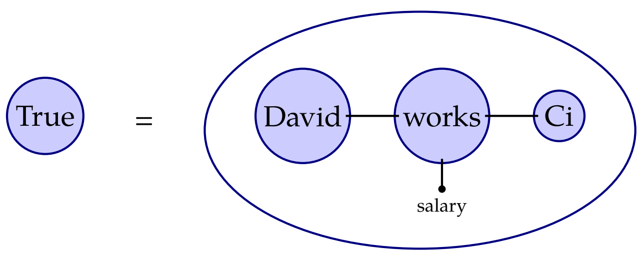
Problem: Assume the following.

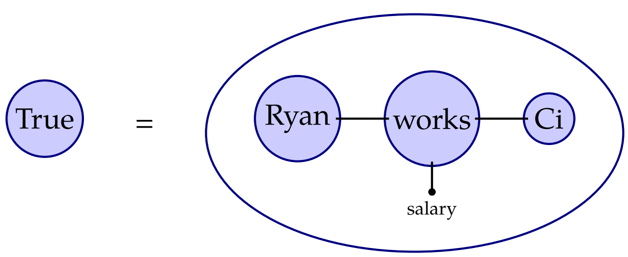


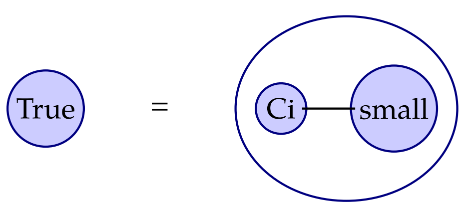
$$\exists(s_1, s_2 : \text{salary})(c : \text{company}).\, \text{small}(c) \wedge \text{works}(p_1, s_1, c) \wedge \text{works}(p_2, s_2, c) \vdash_{p_1, p_2 : \text{person}} \text{acquainted}(p_1, p_2)$$



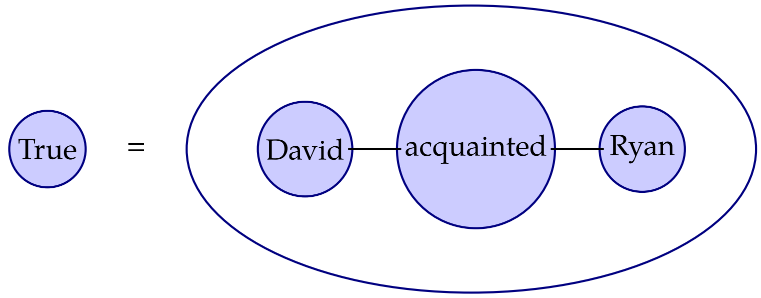
and same for Ryan and Ci







Show that



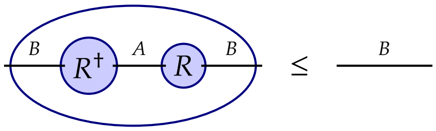
Characterization of morphisms as relations in regular categories: $R \subseteq A \times B$ is the graph of a morphism $A \to B$ iff

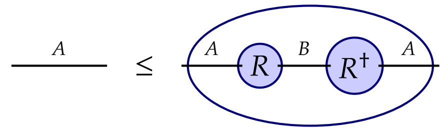
1. composing with the projection $R \subseteq A \times B \to A$ is a regular epimorphism.
2. The square below is a pullback square

$$\begin{array}{ccc} R & \xrightarrow{\Delta_R} & R \times R \\ {\scriptstyle \Delta_B}\downarrow & \lrcorner & \downarrow \\ A \times B \times B & \xrightarrow[\Delta_A]{} & A \times B \times A \times B \end{array}$$

   such that the top map is the diagonal and the lefthand map is the composite $R \subseteq A \times B \xrightarrow{\Delta_B} A \times B \times B$

Exercise: show that the following are equivalent for R associated to $[A, B]$ in the hypergraph world, both say R is a "morphism from A to B":





is equivalent to: