Discrete temporal type theory

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I. Introduction

   A. Goal: a higher-order logic for behavior

      1. Continuous version (joint with P. Schultz, https://arxiv.org/abs/1710.10258)

         a. ODEs, LTSs, delays

         b. Combining disparate sorts of systems

         c. A language for behavior contracts

      2. Discrete version

         a. Simpler, but still quite rich

         b. Kinda like: A higher-order logic for graphs

         c. One can reason about restrictions on paths (e.g. "Whenever g traverses a blue edge, it must traverse two more consecutive blue edges within five hops.")

         d. One can reason about "effects" of traversing longer paths, i.e. information which can't be reduced to what's observable on the edges.

   B. Formal language

      1. Very useful for defining and proving properties about behavior.

      2. Higher-order logic with topos semantics works well.

   C. Plan:

      1. Describe the topos externally

      2. Explain the type theory

      3. Return to the above statement re: graphs

II. Topos of discrete behavior types $\mathcal{B}_{\mathbb{Z}}$

   A. Two presheaf toposes

      1. Geometric theory of discrete finite intervals

         a. For each $d, u \in \mathbb{Z}$ with $d \leq u$, a proposition "$d \leq t \leq u$"[1]

         b. Axiom: $\vdash \bigvee_{d \leq u} d \leq t \leq u$.

[1] "d is for down, u is for up"

c. If $d' \leq d \leq u \leq u'$ then $d \leq t \leq u \vdash d' \leq t \leq u'$

2. Its syntactic category: the open sets of a topological space $\mathbb{IZ}$
   a. Points of $\mathbb{IZ}$ are intervals $[a, b]$ with $a \leq b$
   b. Basis of open subsets: $\{\downarrow[d, u] \mid d \leq u\}$
      (1) I.e. an open set $\downarrow[d, u] = \{[a, b] := d \leq a \leq b \leq u\}$ for each pair of integers $d \leq u$
      (2) Opens are arbitrary unions of these. We'll meet them in some form later.
3. $\mathsf{Psh}(\mathbb{IZ})$
   a. Formal colimit completion of $\mathbb{IZ}$
   b. Has finite limits, nno, exponential objects, subobject classifier
   c. Epi-mono factorization, quotients by equivalence relations, disjoint coproducts
4. $\mathbb{Z}$-action and quotient topos
   a. For any $n \in \mathbb{Z}$ and open $[d, u] \in \mathbb{IZ}$, have $[d + n, u + n]$
   b. For any $X \in \mathsf{Psh}(\mathbb{IZ})$, let $T(X)[d, u] := \prod_{n \in \mathbb{Z}} X[d + n, u + n]$
   c. T is a left-exact comonad. Denote topos of coalgebras by $\mathsf{Psh}(\mathbb{IZ})_{\mathbb{Z}}$.
   d. Let $\mathbf{Int}_{\mathbb{Z}}$ denote localization of $\mathbb{IZ}$ by $\mathbb{Z}$-action:

$$\mathrm{Ob}(\mathbf{Int}_{\mathbb{Z}}) = \{[d, u] \mid d \leq u\}$$
$$\mathbf{Int}_{\mathbb{Z}}([d, u], [d', u']) = \{n \in \mathbb{Z} \mid [d + n, u + n] \subseteq [d', u']\}$$

   e. As always, $\mathbf{Int}_{\mathbb{Z}}$ is equivalent to its skeleton
      (1) Formally: $\mathbf{Int}_{\mathbb{Z}}$ is twisted arrow category of $\mathbb{N}$, as a one-object cat
      (2) Concretely:

$$\mathrm{Ob}(\mathbf{Int}_{\mathbb{Z}}) = \mathbb{N}$$
$$\mathbf{Int}_{\mathbb{Z}}(\ell', \ell) = \{n \in \mathbb{N} \mid n + \ell' \leq \ell\}$$

   f. Theorem: $\mathsf{Psh}(\mathbf{Int}_{\mathbb{Z}}) \cong \mathsf{Psh}(\mathbb{IZ})_{\mathbb{Z}}$, call it $\mathcal{B}_{\mathbb{Z}}$

B. Examples in $\mathcal{B}_{\mathbb{Z}} = \mathsf{Psh}(\mathbf{Int}_{\mathbb{Z}})$:

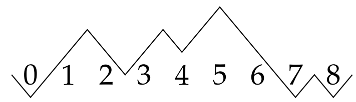
1. Graphs (fully faithful)
2. Representables $\mathsf{y}\ell$ given by $\mathsf{y}\ell(\ell') = \mathbf{Int}_{\mathbb{Z}}(\ell', \ell) = \{n \in \mathbb{N} \mid n + \ell' \leq \ell\}$
3. Simplicial sets (faithful, not full), induced by "obvious functor" $\mathbf{Int} \to \Delta$
4. Presheaf $X(0) := \{V\}$, $X(1) := \{E\}$, $X(2) := \{P_1, P_2\}$, $X(n + 3) := \emptyset$.

C. The subobject classifier and Catalan numbers

1. Calculate $\Omega(\ell)$ for $\ell = 0, 1$

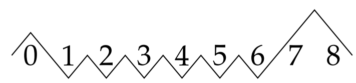
2. 2, 5, 14, 42, 132, ... Catalan numbers
3. Dyck paths as subobject classifier

$P_0 :=$



4. the poset
   a. order $\leq$ is "always below";
   b. meet $\wedge$ is min, wedge $\vee$ is max,
   c. Heyting structure: $P \Rightarrow Q$ is supremum of R with $P \wedge R \leq Q$.
5. Example: $\neg P_0$.

$\neg P_0 =$



III. Topos-theoretic type theory

A. My take on type theory

1. Introduce some atomic types, function symbols, predicate symbols
2. Topos-theoretic rules:
   a. products, coproducts, function types, subtypes, quotient types
   b. Natural numbers type $\mathbb{N}$, type of propositions `Prop`
   c. Predicate symbols can be combined using $\top, \wedge, \bot, \vee, \Rightarrow, \exists, \forall$
3. Add axioms
4. Prove things from your axioms

B. Topos semantics in a presheaf topos $\mathcal{E} = \mathsf{Psh}(C)$

1. Assignments
   a. To each atomic type, assign an object in $\mathcal{E}$
   b. To each atomic function symbol, assign a morphism in $\mathcal{E}$
2. Mitchell-Bénabou language
   a. Want to check that each axiom holds; what do connectives and quantifiers mean?
   b. Connectives given by definable maps $\Omega^k \to \Omega$ for various $k \in \mathbb{N}$
   c. Quantification over X is given by definable maps $\Omega^X \to \Omega$.
3. Kripke-Joyal semantics of predicate $P : X \to$ `Prop` ... **in a presheaf topos!** $\mathsf{Psh}(C)$
   a. Notation
      (1) Take $c \in C$ in the site and a section $x \in X(c)$

(2) Write $c \Vdash P(x)$ to mean that $\mathsf{y}c \xrightarrow{x} X$ factors through $\{X \mid P\} \subseteq X$.

b. Then it turns out that for any $x \in X(c)$:

(1) $c \Vdash P(x) \wedge Q(x)$ iff $c \Vdash P(x)$ and $c \Vdash Q(x)$.

(2) $c \Vdash P(x) \vee Q(x)$ iff $c \Vdash P(x)$ or $c \Vdash Q(x)$.

(3) $c \Vdash P(x) \Rightarrow Q(x)$ iff the following holds for all $d \to c$ in C: if $d \Vdash P(x|_d)$ then $d \Vdash Q(x|_d)$.

(4) $c \Vdash \neg P(x)$ iff for each $d \to c$ in C, it is *not the case that* $d \Vdash P(x|_d)$

(5) $c \Vdash \forall(y : Y).\, P(x, y)$ iff, for all $d \to c$ in C and all $y \in Y(d)$, we have $d \Vdash P(x|_d, y)$.

(6) $c \Vdash \exists(y : Y).\, P(x, y)$ iff there exists $y \in Y(c)$ with $c \Vdash P(x, y)$.

c. Warning: the above facts only hold for presheaf toposes.

IV. Temporal type theory

A. A type theory with semantics in $\mathcal{B}_{\mathbb{Z}}$

1. Atomic type $\texttt{Time}$, atomic predicate $\delta\colon \mathbb{Z} \times \texttt{Time} \to \texttt{Prop}$

2. Some axioms:

a. $\forall(n : \mathbb{Z})(t : \texttt{Time}).\, \neg\neg\delta(n, t) \Rightarrow \delta(n, t)$.

(1) Let $\upsilon\colon \mathbb{Z} \times \texttt{Time} \to \texttt{Prop}$ be $\upsilon(n, t) := \neg\delta(n + 1, t)$

(2) Syntactic sugar: Write $n \leq t$ for $\delta(n, t)$, and write $t \leq n$ for $\upsilon(n, t)$

(3) We have $t \leq n$ iff $\neg(n + 1 \leq t)$

b. $\forall(t : \texttt{Time}).\, \exists(n : \mathbb{Z}).\, n \leq t$

c. $\forall(t : \texttt{Time}).\, \exists(n : \mathbb{Z}).\, t \leq n$

d. $\forall(t : \texttt{Time})(m, n : \mathbb{Z}).\, (m \leq n) \wedge (n \leq t) \Rightarrow (m \leq t)$

e. $\forall(t_1, t_2 : \texttt{Time}).\, \big(\forall(n : \mathbb{Z}).\, (n \leq t_1) \Leftrightarrow (n \leq t_2)\big) \Rightarrow (t_1 = t_2)$

f. Torsor axioms:

(1) $\forall(t : \texttt{Time})(n' : \mathbb{Z}).\, \exists(t' : \texttt{Time}).\, \forall(n : \mathbb{Z}).\, \big((n + n' \leq t') \Leftrightarrow (n \leq t)\big)$

(2) Given t, n', write $(t + n') : \texttt{Time}$ for unique such t'

(3) $\forall(t, t' : \texttt{Time}).\, \exists(n' : \mathbb{Z}).\, \forall(n : \mathbb{Z}).\, \big((n + n' \leq t') \Leftrightarrow (n \leq t)\big)$

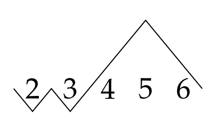
(4) Given t, t', write $(t' - t) : \mathbb{Z}$ for unique such n'

B. The semantics

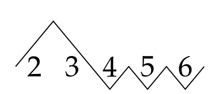
1. Atomic type $\texttt{Time}$ is sent to the graph $\cdots \to \bullet^{-1} \to \bullet^{0} \to \bullet^{1} \to \bullet^{2} \to \cdots$

2. Predicate $\delta\colon \mathbb{Z} \times \texttt{Time} \to \texttt{Prop}$ has $\ell \vdash (n, [t_0, t_1, \ldots, t_\ell])$ iff $n \leq t_0$.

$(4 \leq [2, 3, 4, 5, 6]) :=$



$\neg(4 \leq [1, 2, 3, 4, 5, 6]) :=$



C. Useful modalities

1. A modality $j\colon \texttt{Prop} \to \texttt{Prop}$ is a function satisfying the following for all $P, Q : \texttt{Prop}$

   a. $P \Rightarrow jP$,

   b. $jjP \Rightarrow P$, and

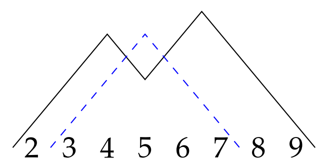
   c. $j(P \wedge Q) \Leftrightarrow (jP \wedge jQ)$.

2. $\downarrow\colon \texttt{Time} \to (\mathbb{Z} \times \mathbb{Z}) \to \texttt{Prop} \to \texttt{Prop}$

   a. Write $t \,\#\, [d, u]$ to mean $(t \le d - 1) \vee (u + 1 \le t)$, "$t$ is apart from $[d, u]$"

   b. Define $\downarrow^t_{[d,u]} P := P \vee t \,\#\, [u, d]$.[2]

   c. Example $\downarrow^t_{[3,7]} \bot = t \,\#\, [7, 3] = (t \le 6) \vee (4 \le t)$, with $t = [2, \ldots, 9]$:

$\downarrow_{[3,7]}(\bot) =$ 

   d. $\downarrow_{[d,u]} P$ wipes all information about P except what occurs on intervals containing $[d, u]$.

3. $@\colon \texttt{Time} \to (\mathbb{Z} \times \mathbb{Z}) \to \texttt{Prop} \to \texttt{Prop}$

   a. Define $@^t_{[d,u]} P := (P \Rightarrow t \,\#\, [u, d]) \Rightarrow t \,\#\, [u, d]$

   b. $@^t_{[d,u]} P$ wipes all information about P except what occurs on the interval $[d, u]$.

   c. Derive modality "$\mathrm{At}_{[d,u]} P \rightsquigarrow$ " given by

$$\mathrm{At}_{[d,u]} P \rightsquigarrow Q := (@_{[d,u]} P) \Rightarrow \downarrow_{[d,u]} Q.$$

4. $\pi\colon \texttt{Prop} \to \texttt{Prop}$, "pointwise"

   a. Defined by $\epsilon P := \forall (t : \texttt{Time}).\, @^t_{[0,0]} P$

   b. The subtopos defined by π is a subtopos of sets.

5. $\epsilon\colon \texttt{Prop} \to \texttt{Prop}$, "edgewise"

   a. Defined by $\epsilon P := \forall (t : \texttt{Time}).\, @^t_{[0,1]} P$

   b. The subtopos defined by ϵ is a subtopos of graphs.

D. Finally, we have nice language

1. Example: given a graph G and a subgraph $B \subseteq G$ defined by $i_B\colon G \to \texttt{Prop}$

2. $\left\{ g : G \;\middle|\; \forall (t : \texttt{Time}).\, \mathrm{At}_{[-1,0]} i_B(g) \rightsquigarrow \exists (n : \mathbb{Z}).\, 0 \le n \le 5 \wedge @^t_{[n,n+2]} i_B(g) \right\}$.

3. "Whenever g traverses a blue edge, it must traverse two more consecutive blue edges within five hops."

[2] Inverted order not a typo.