Decomposition spaces and toposes

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Joint with Joachim Kock

I. Introduction

A. Decomposition spaces introduced by

1. Kock et al. in combinatorics,
2. Dyckerhoff and Kapranov in homological alg. rep. theory

B. Idea:

1. In a category you can compose
2. In a decomposition space you can decompose
3. All categories are decomp spaces, but not vice versa.
4. The nice morphisms are CULF maps: conservative, unique-lifting-of-factorizations

C. Example: $B(\mathbb{N}, 0, +)$

1. Categories CULF over it are graphs
2. Decomposition spaces CULF over it are discrete behavior types.

D. Our result and its history:

1. In 1996, Lamarche conjectured that categories CULF over $\mathcal{C}$ form a topos.
2. Bunge and Niefield announced a proof
3. Johnstone found an error.
4. We prove the result when categories are replaced by decomp spaces.

E. Theorem: $\mathsf{Decomp}/\mathcal{D}$ is a topos for any decomposition space $\mathcal{D}$.

II. Simplicial background

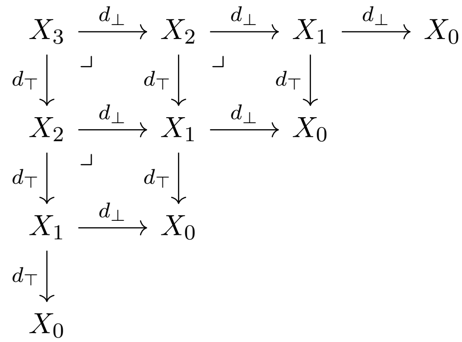
A. Definition of simplicial set

1. Faces and degeneracies
2. Active and inert (factorization system)

B. Examples

1. $\Delta^{-}\colon \mathbb{\Delta} \to \mathsf{Top}$ and $i\colon \mathbb{\Delta} \to \mathsf{Cat}$.
2. These induce singular complex and nerve
   a. $N\colon \mathsf{Cat} \to \mathsf{sSet}$ is fully faithful

b. A simplicial set is a nerve iff it satisfies Segal condition:



"Top and bottom face maps pullback along each other."

C. Category of elements, discrete fibrations

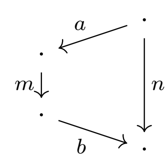
1. $\int\colon \mathsf{Psh}(C) \to \mathsf{Cat}_{/C}$, write $\int(X) =: (\mathrm{el}(X) \xrightarrow{\pi} C)$
2. Also have $\partial\colon \mathsf{Cat}_{/C} \to \mathsf{Psh}(C)$
3. $\partial \int = \mathrm{id}$.
4. $\int(X)$ is always a discrete-fibration; $(\partial, \int)$ form equivalence on discrete fibs.

D. Edgewise subdivision $Q\colon \mathbb{\Delta} \to \mathbb{\Delta}$ and twisted arrows.

1. Segal's subdivision:

$$Q[n] := [n]^{\mathrm{op}} \vee [n] = [2n{+}1], \qquad \begin{array}{l} 0 \longrightarrow 1 \longrightarrow \cdots \longrightarrow n \\ \uparrow \\ 0' \longleftarrow 1' \longleftarrow \cdots \longleftarrow n' \end{array}$$

2. For a category X, $Q^*(NX) = N(\mathrm{tw}\, X)$.
3. Example: $\mathrm{tw}(B\mathbb{N})$ has objects $\mathbb{N}$ and morphisms $m \to n$,



III. Decomposition spaces and CULF maps

A. Definition of decomposition space

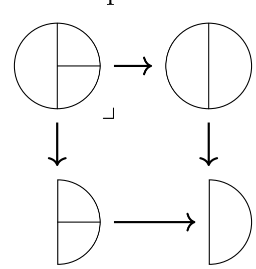
1. Actives and inerts pushout; ask X to take these to pullbacks.

B. Idea: want a coalgebra $V \to V \otimes V$, $V \to 1$

1. Here V has basis X_1.
2. Use $X_1 \xleftarrow{d_1} X_2 \xrightarrow{(d_0,d_2)} X_1 \times X_1$ and $X_1 \xleftarrow{s_0} X_0 \xrightarrow{!} 1$.

C. Examples

1. Graphs with vertex set split into k parts



a. Here $V \in \mathsf{Vect}$ has basis the set of graphs.
b. The comultiplication map $V \to V \otimes V$ sends $[G] \mapsto \sum_{G \rightsquigarrow G_1 + G_2} [G_1] \otimes [G_2]$

2. Categories.

D. CULF maps: natural transformations, cartesian along active maps

E. Twist Q^*X of decomposition space X is category, of CULF is discrete fib

1. Check Segal condition for Q^*X: top and bottom maps pullback.
2. Top maps in Q^*X are removal of n, n'; bottom maps are removal of $0, 0'$.
3. Tops inert, bots active. Since X is decomp, these pullback.
4. Similarly, $Q^*(f)$ on bottom face maps is f on "remove $0, 0'$", active.

IV. Theorem and example

A. Statement: an equivalence $\mathsf{Decomp}_{/\mathcal{D}} \simeq \mathsf{Psh}(\mathrm{tw}\,\mathcal{D})$.

B. One direction $\mathsf{Decomp}_{/\mathcal{D}} \to \mathsf{Psh}(\mathrm{tw}\,\mathcal{D})$: twist

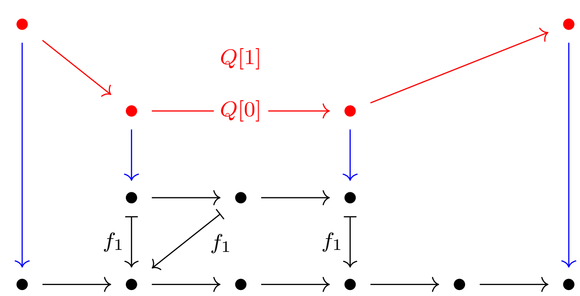
C. Other direction: "untwist"

1. Natural transformation $\mathsf{sSet} \underset{Q^*}{\overset{N \circ \mathrm{el}}{\rightrightarrows}} \mathsf{sSet}$, $\Downarrow \Lambda$

Unique:

$$\begin{array}{ccccccccc} Q[0] & \xrightarrow{Q(d^\top)} & Q[1] & \xrightarrow{Q(d^\top)} & \cdots & \xrightarrow{Q(d^\top)} & Q[k] & & \\ {\scriptstyle \alpha(n_0)}\downarrow & & {\scriptstyle \alpha(f_1)}\downarrow & & & & {\scriptstyle \alpha(f_k)}\downarrow & \overset{\Lambda(f_1,\ldots,f_k,\sigma)}{\searrow} & \\ [n_0] & \xrightarrow[f_1]{} & [n_1] & \xrightarrow[f_2]{} & \cdots & \xrightarrow[f_k]{} & [n_k] & \xrightarrow[\sigma]{} & X \end{array}$$

This can be imagined as follows (take $k = 1$, $n_0 = 2$, $n_1 = 5$):



2. Λ restricts to functor λ: $\mathrm{el}(\mathcal{D}) \to \mathrm{tw}(\mathcal{D})$ on decomposition spaces
3. Define $\mathrm{untw}(p) := \partial p' = \partial(\lambda_{\mathcal{D}}^* p)$

$$\begin{array}{ccc} F' & \longrightarrow & F \\ {\scriptstyle p'}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ \mathrm{el}(\mathcal{D}) & \xrightarrow[\lambda_{\mathcal{D}}]{} & \mathrm{tw}(\mathcal{D}) \end{array}$$

D. Example: $\mathcal{D} = B(\mathbb{N}, 0, +)$.

1. $\mathsf{Psh}(\mathrm{tw}\,\mathcal{D})$ = "Discrete temporal type theory"

2. Given $F\colon \operatorname{tw}(\mathcal{D})^{\mathrm{op}} \to \mathsf{Set}$, have

$$(\operatorname{untw} F)_k = \coprod_{a_1,\cdots,a_k \in \mathbb{N}} F(a_1 + \cdots + a_n)$$

3. This maps CULF to $\mathcal{D}$, since

$$\mathcal{D}_k = \coprod_{a_1,\cdots,a_k \in \mathbb{N}} 1$$

   and $\mathcal{D}$ on active maps preserves sums: $\sum a_i = \sum b_j$.
4. Idea: $(\operatorname{untw} F)_k$ records k-fold decompositions of F-behaviors.