Double categories

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Abstract

Abstract: I will talk briefly about models of limit sketches in various categories, ending up with double categories.

I. Introduction

A.

II. Limit sketches

A. Basic idea

B. Cones in categories

1. Cone indexing categories: small categories with distinguished initial object
   a. Cone point: initial object
   b. Rest: full subcategory spanned by everything else.
2. Cone diagrams in $\mathcal{C}$: functors to $\mathcal{C}$

C. Limit sketches

1. Limit sketch $(\mathcal{C}, K)$: a category $\mathcal{C}$ with some set K of cone diagrams
2. Sketch morphism: functor sending limit cones to limit cones
   i.e. $F\colon \mathcal{C} \to \mathcal{C}'$ plus a function $f\colon K \to K'$ such that $F \circ k = f(k)$ for all $k \in K$.

D. Models

1. Some categories $\mathcal{C}$ have canonical cones: namely when $\mathcal{C}$ has limits.
   a. For every diagram with limit L, make L a chosen cone.
2. Examples:
   a. **Set** has all small limits: make these the chosen cones
   b. **Cat** has all small limits: make these the chosen cones
3. $\mathcal{S}$-models of a limit sketch $(\mathcal{C}, K)$ are sketch morphisms to a category $\mathcal{S}$ with limits.

III. Examples of limit sketches and models

A. Magmas

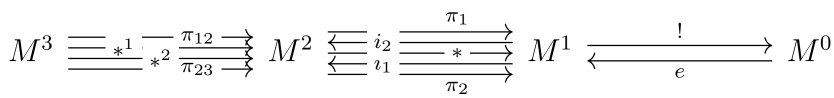
1. Sketch $(\mathcal{C}, K)$.
   a. $\mathcal{C} = \boxed{M^2 \overset{\pi_1}{\underset{\pi_2}{\overset{*}{\rightrightarrows\!\!\to}}} M}$
   b. One limit cone, $\boxed{\begin{array}{ccc} M^2 & \xrightarrow{\pi_1} & M \\ {\scriptstyle \pi_2}\downarrow & & \\ M & & \end{array}}$
2. Models in **Set**: sets X with binary operation $X \times X \to X$.
3. Models in **Cat**: categories $\mathcal{C}$ with functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$.
4. Models in **Top**: topological spaces T with continuous map $T \times T \to T$.

B. Pointed objects

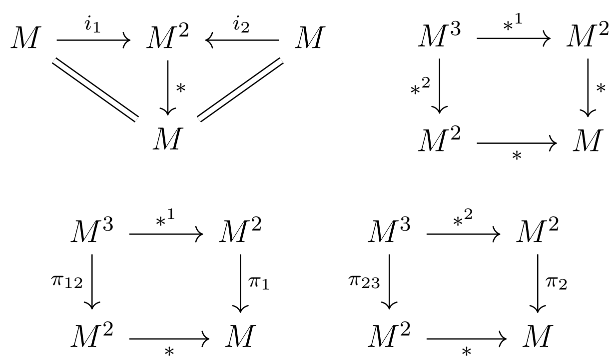
1. Sketch $(\mathcal{C}, K)$.
    a. $\mathcal{C} = \boxed{M^0 \xrightarrow{p} M}$
    b. One limit cone, $\boxed{M^0}$
2. Models in **Set**: sets X with chosen point $p \in X$.
3. Models in **Cat**: categories $\mathcal{C}$ with chosen object $c \in \mathcal{C}$.
4. Models in **Top**: topological spaces T with chosen point $t \in T$.

C. Monoids (combine the above, and add laws)

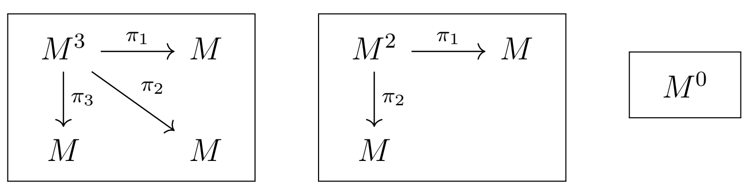
1. Sketch $(\mathcal{C}, K)$.
    a. $\mathcal{C}$ has four objects...



with commutative diagrams such as



    b. K includes three limit cones:



2. Models: "monoid objects in ..."
    a. in **Set** are monoids.
    b. in $\top$ are "topological monoids," like $\mathbb{R}$ or the circle
    c. in **Cat**: (strict) monoidal categories:
        (1) A category $\mathcal{C}$,
        (2) a functor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, and
        (3) a functor $I\colon 1 \to \mathcal{C}$
        (4) such that $I \otimes c = c = c \otimes I$, etc.

d. in monoids is: a commutative monoid.

$m * e = m = e * m$.

homomorphism implies $(m_1 * m_2) \cdot (m_1' * m_2') = (m_1 \cdot m_1') * (m_2 \cdot m_2')$, which implies $m_1 \cdot m_2' = (m_1 * e) \cdot (e * m_2') = (m_1 \cdot e) * (e \cdot m_2') = m_1 * m_2'$. Thus $* = \cdot$.

Finally $m_2 * m_1' = (e * m_2) \cdot (m_1' * e) = (e \cdot m_1') * (m_2 \cdot 2) = m_1' * m_2$, so commutative.

D. Categories

1. Sketch $(\mathcal{C}, K)$.

   a. $\mathcal{C}$ has four objects C_3, C_2, C_1, C_0, two maps $dom, cod \colon C_1 \rightrightarrows C_0$, three maps $L, R, \circ \colon C_2 \to C_1$, etc.

   b. K consists of two cones

$$\begin{array}{ccc} C_2 & \xrightarrow{R} & C_1 \\ {\scriptstyle L}\downarrow & \lrcorner & \downarrow{\scriptstyle dom} \\ C_1 & \xrightarrow[cod]{} & C_0 \end{array} \qquad \begin{array}{ccc} C_3 & \xrightarrow{RR} & C_2 \\ {\scriptstyle LL}\downarrow & \lrcorner & \downarrow{\scriptstyle L} \\ C_2 & \xrightarrow[R]{} & C_1 \end{array}$$

2. Models: "category objects in..."

   a. in **Set** are categories.

   b. in monoids: (strict) monoidal categories:

      (1) monoids C_1, C_0 (tensor objects, tensor morphisms),

      (2) monoid homomorphisms $dom, cod \colon C_1 \rightrightarrows C_0$ $((dom(f \otimes g) = dom(f) \otimes dom(g))$

      (3) monoid homomorphism $\circ \colon C_1 \times_{C_0} C_1 \to C_1$

      monoid homomorphism condition means $(f \otimes f') \circ (g \otimes g') = (f \circ g) \otimes (f' \circ g')$.

   c. in **Cat**: double categories.

IV. Double categories

A. A double category $\mathbb{D}$ is a category object in categories.

1. It consists of categories $\mathbb{D}_0, \mathbb{D}_1$ and functors $dom, cod \colon \mathbb{D}_1 \rightrightarrows \mathbb{D}_0$,
2. Plus an associative composition $\circ \colon \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \to \mathbb{D}_1$.

B. Making sense of this

1. Vertical category $\mathbb{D}_0$

   a. Denote objects of $\mathbb{D}_0$ as objects

   b. Denote morphisms of $\mathbb{D}_0$ as vertical morphisms.

2. Category $\mathbb{D}_1$

   a. Each object of $R \in \mathbb{D}_1$ has $dom(R), cod(R) \in \mathbb{D}_0$, objects.

   b. Denote it $A \xrightarrow{R} B$, if $dom(R) = A, cod(R) = B$.

c. Denote morphisms of $\mathbb{D}_1$ as lax-squares

$$\begin{array}{ccc} A & \xrightarrow{R} & B \\ {\scriptstyle f}\downarrow & \Downarrow & \downarrow{\scriptstyle g} \\ C & \xrightarrow[S]{} & D \end{array}$$

d. Composition in $\mathbb{D}_1$ is pasting squares vertically.

3. The extra composition $\circ\colon \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \to \mathbb{D}_1$ is horizontal.

$$\begin{array}{ccccc} A & \xrightarrow{R} & B & \xrightarrow{R'} & C \\ {\scriptstyle f}\downarrow & \Downarrow & \downarrow{\scriptstyle g} & \Downarrow & \downarrow{\scriptstyle h} \\ A' & \xrightarrow[S]{} & B' & \xrightarrow[S']{} & C' \end{array}$$

And the fact that $\circ$ is a functor means horizontal and vertical composition interchange.

C. Examples

1. From a category C.
   a. Commutative squares in a category.
   b. Horizontals are all maps, verticals are isomorphisms
   c. if C has pullbacks, take vertical maps as C and horizontal maps as $span(C)$
2. Categories, functors, and profunctors
3. We may see more of these in future talks if Brendan and my stuff works out.