# Some categorical perspectives on institutions

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## 1 Reformulating the usual definition

For any category $\mathcal{S}$, there are equivalences of categories

$$\mathsf{Fun}(\mathcal{S}, \mathsf{Set}^{\mathrm{op}}) \cong \mathsf{DFib}_{\mathcal{S}} \qquad \text{and} \qquad \mathsf{Fun}(\mathcal{S}, \mathsf{Set}) \cong \mathsf{DoFib}_{\mathcal{S}}. \tag{1}$$

where $\mathsf{DFib}_{\mathcal{S}}$ is the category of discrete fibrations over $\mathcal{S}$ and $\mathsf{DoFib}_{\mathcal{S}}$ is the category of discrete opfibrations over $\mathcal{S}$.

**Definition 1.1.** An *institution base* consists of a category $\mathcal{S}$ (*signatures*), a discrete opfibration $\lambda\colon \mathcal{N} \to \mathcal{S}$ (*sentences*), and a discrete fibration $\rho\colon \mathcal{M} \to \mathcal{S}$ (*models*).

$$\mathcal{N} \xrightarrow{\lambda} \mathcal{S} \xleftarrow{\rho} \mathcal{M}$$

Given $f\colon s \to s'$, and $n \in \mathcal{N}$ with $\lambda(n) = s$, write $n \to f_!(n)$ for the (unique) cocartesian map in $\mathcal{N}$; similarly, given $m' \in \mathcal{M}$ with $\rho(m') = s'$, write $f^*(m') \to m'$ for the (unique) cartesian map in $\mathcal{M}$.

An *$S$-indexed satisfaction relation* consists of a subset $\Dashv_s \subseteq \lambda^{-1}(s) \times \rho^{-1}(s)$ for each $s \in S$ such that, for each $f\colon s \to s'$ with $n \in \rho^{-1}(s)$ and $m' \in \rho^{-1}(s')$, we have

$$n \Dashv_s f^*(m') \quad \text{iff} \quad f_!(n) \Dashv_{s'} m' \tag{2}$$

where we write $n \Dashv_s m$ to denote $(n, m) \in \Dashv_s$.

An *institution* $(\mathcal{S}, \mathcal{M}, \rho, \mathcal{N}, \lambda, \Dashv)$ consists of an institution base together with an $\mathcal{S}$-indexed satisfaction relation.

*Remark* 1.2. The above definition of institution is only cosmetically different from the usual one. The usual one considers a category $\mathsf{Sign}$ and a pair of functors $\mathrm{Sen}\colon \mathsf{Sign} \to \mathsf{Set}$ and $\mathrm{Mod}\colon \mathsf{Sign} \to \mathsf{Set}^{\mathrm{op}}$. In **??**, $\mathsf{Sign}$ is replaced with $\mathcal{S}$, and the two set-valued functors are replaced by a discrete opfibration $\lambda$ and a discrete fibration $\rho$; this is equivalent by **??**. We also switch the direction of the $\vDash$ symbol, to make the coherence equation (**??**) look more familiar to category theorists.

*Remark* 1.3. The discrete fibrations and opfibrations in **??** can be replaced by non-discrete versions, for an equally interesting notion of institution.

**Proposition 1.4.** *Both discrete fibrations and discrete opfibrations are stable under pullback:*

$$\begin{array}{ccc} \mathcal{P}' & \longrightarrow & \mathcal{P} \\ {\scriptstyle F'}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle F} \\ \mathcal{S}' & \underset{G}{\longrightarrow} & \mathcal{S} \end{array}$$

*i.e. if $G$ is any functor and $F$ is a discrete fibration or a discrete opfibration, then so is $F'$*

**Proposition 1.5.** *An institution can be identified with a commutative square of categories*

$$\begin{array}{ccccc} & & \mathcal{I} & & \\ & {\scriptstyle \rho'}\swarrow & & \searrow{\scriptstyle \lambda'} & \\ \mathcal{N} & & & & \mathcal{M} \\ & {\scriptstyle \lambda}\searrow & & \swarrow{\scriptstyle \rho} & \\ & & \mathcal{S} & & \end{array} \tag{3}$$

*such that*

1. *$\lambda$ and $\lambda'$ are discrete opfibrations,*
2. *$\rho$ and $\rho'$ are discrete fibrations, and*
3. *the induced map $\mathcal{I} \to \mathcal{M} \times_{\mathcal{S}} \mathcal{N}$ is a monomorphism.*

*It will follow automatically that $\mathcal{I} \to \mathcal{M} \times_{\mathcal{S}} \mathcal{N}$ is the inclusion of a full subcategory.*

*Proof.* Given $\mathcal{I} \subseteq \mathcal{N} \times_{\mathcal{S}} \mathcal{M}$, define $\dashv_s$ for any $s \in \mathcal{S}$ on a given pair $(n, m) \in (\lambda^{-1}(s) \times \rho^{-1}(s) = \mathcal{N} \times_{\mathcal{S}} \mathcal{M}$ by writing $n \dashv_s n$ iff $(n, m) \in \mathcal{I}$. We need to show that it satisfies the coherence condition (**??**), so take any $f \colon s \to s'$ with $n \in \rho^{-1}(s)$ and $m' \in \rho^{-1}(s')$. First suppose $n \dashv_s f^*(m')$. Then $i \coloneqq (n, f^*(m')) \in \mathcal{I}$ and we have a cartesian map $\lambda'(i) = f^*(m') \to m'$ over $f$, so we get a cocartesian lift $f'' \colon i \to i'$ where $\lambda(i') = m'$. We want to show that $i'$ represents $f_!(n) \dashv m'$, and it suffices to show $\rho'(i') = f_!(n)$. Since $f = \rho(\lambda'(f'')) = \lambda(\rho'(f''))$, this follows from the uniqueness of cocartesian lifts $n \to n'$ over $f$. The argument that $f_!(n) \dashv m'$ implies $n \dashv f^*(m')$ is similar.

Now suppose that one has an institution $(\mathcal{S}, \mathcal{M}, \rho, \mathcal{N}, \lambda, \vDash)$; define $\mathcal{I}$ to be the full subcategory of $\mathcal{M} \times_{\mathcal{S}} \mathcal{N}$ spanned by the objects $(n, m)$ for which $n \dashv m$. We need to show that $\lambda'$ (obtained by restricting the projection $\mathcal{N} \times_{\mathcal{S}} \mathcal{M} \to \mathcal{M}$) is a discrete opfibration; showing that $\rho'$ is a discrete fibration is similar. Suppose given $(n, m) \in \mathcal{I}$ and a map $f' \colon m \to m'$ in $\mathcal{M}$; we need to find a cocartesian lift $f'' \colon (n, m) \to (n', m')$ over it, for some $n'$. All maps in $\mathcal{M}$ are cartesian, so there is some $f \colon \rho(m) \to \rho(m')$ over which $f'$ is cartesian, i.e. $m = f^*(m')$, and we have a cocartesian lift $n \to f_!(n)$ in $\mathcal{N}$; take $n' \coloneqq f_!(n)$. By definition, $n \dashv m$ holds and implies $n' \dashv m'$, so $(n', m') \in \mathcal{I}$. Since $\mathcal{I} \subseteq \mathcal{M} \times_{\mathcal{S}} \mathcal{N}$ is full, it contains the required morphism. □

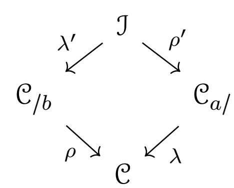
**Proposition 1.6** (Remy Tuyeras's reformulation)**.** *An institution can be identified with a category $\mathcal{S}$, functors $N \colon \mathcal{S} \to \mathsf{Set}$ and $M \colon \mathcal{S}^{\mathrm{op}} \to \mathsf{Set}$, and a subset $I$ of the coend $\int^{s \in \mathcal{S}} N(s) \times M(s)$.*

*Proof.* A subset of $\int^{\mathcal{S}} N(s) \times M(s)$ can be identified with a function $I\colon \left(\int^{\mathcal{S}} N(s) \times M(s)\right) \to \{\top, \bot\}$. Since a coend is a universal co-wedge, this can in turn be identified with a function $I_s\colon N(s) \times M(s) \to \{\top, \bot\}$ for each $s \in \mathcal{S}$, such that for all $s, s' \in \mathcal{S}$, the following diagram commutes:

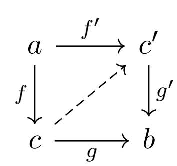
$$\begin{CD}
N(s) \times \mathcal{S}(s,s') \times M(s') @>{N(s)\times M(-)}>> N(s) \times M(s) \\
@V{N(-)\times M(s')}VV @VV{I(s)}V \\
N(s') \times M(s') @>>{I(s')}> \{\top, \bot\}
\end{CD}$$

This in turn can be identified with a subset $I_s \subseteq N(s) \times M(s)$ such that for each $f\colon s' \to s$ in $\mathcal{S}$ and elements $n \in N(s)$ and $m' \in M(s')$ we have $I_s(m', N(f)(n))$ iff $I_{s'}(M(f)(m'), n)$. This is the definition of institution from **??**. □

*Example* 1.7. Let $\mathcal{C}$ be any category, choose objects $a, b \in \mathcal{C}$, and choose a subset $S \subseteq \mathcal{C}(a,b)$ of the hom-set. We can create an institution from this data as follows. Use the slice category over $b$ as the models, $\mathcal{M} := \mathcal{C}_{/b}$, where $\rho(c \to b) := c$. Similarly, use the coslice under $a$ as the sentences, $\mathcal{N} := \mathcal{C}_{a/}$, where $\lambda(a \to c) := c$. We still need to define $\mathcal{I}$, $\lambda'$, and $\rho'$ in the diagram below.




Define $\mathcal{I}$ to have as objects $\{(a \xrightarrow{f} c \xrightarrow{g} b) \mid g \circ f \in S\}$ and as morphisms all maps $c \to c'$ making the obvious diagram commute:




Take $\lambda'(f,g) = f$ and $\rho'(f,g) = g$. It is easy to check that $\lambda'$ is a discrete opfibration and that $\rho'$ is a discrete fibration.

*Example* 1.8. Let $\pi\colon \mathcal{E} \to \mathcal{B}$ be a posetal bifibration, i.e. the fiber $\pi^{-1}(b)$ is a poset for every $b \in \mathcal{B}$, and $\pi$ is both a fibration and an opfibration. In particular, the cocartesian map and the cartesian map over any given $f\colon b \to b'$ in $\mathcal{B}$ are adjoint. Then $\mathcal{E}$ can be identified with a diagonal subcategory $\mathcal{E} \subseteq \mathcal{E} \times_{\mathcal{B}} \mathcal{E}$, and this gives a (non-discrete) institution because the required fibrations and opfibrations are identities $\mathcal{E} \to \mathcal{E}$.

In the language of institutions, the category of sentences is the same as the category of models, and a sentence $n$ is modeled by a model $m$ iff there is a map $n \to m$; this perhaps gives a reason that **??** looks like an adjunction.

*Example* 1.9. Take $\mathcal{S} \coloneqq \mathsf{Cat}$ and take $\mathcal{M} = \mathcal{N} = \mathsf{DoFib}$, with $\lambda$ and $\rho$ acting on objects by sending a discrete opfibration $E \to B$ to its codomain $B$, but with $\rho$ given by pullback and $\lambda$ given by left Kan extension. Objects in the pullback (not pseudo or lax pullback) $\mathcal{N} \times_{\mathcal{S}} \mathcal{M}$ can be identified with pairs of functors $I, J\colon C \to \mathsf{Set}$ for some category $C \in \mathcal{S}$. We define $\mathcal{I}$ to be those $(I, J)$ for which a natural transformation $I \to J$ exists.

*Example* 1.10. Consider the (underlying 1-category of the) category $\mathsf{Topos}$ of toposes and geometric morphisms; this will serve as the category of signatures. Let $\mathtt{pt}\colon \mathsf{Topos} \to \mathsf{Cat}$ denote the functor that sends each topos to its category of points; the points will serve as the "models" of a topos.

For any topos $\mathcal{E}$, we have a poset $\mathcal{E}(1, \Omega_{\mathcal{E}})$ of truth values; here $\Omega_{\mathcal{E}}$ is the subobject classifier of $\mathcal{E}$, and we can identify a truth value with a subobject $e \subseteq 1$ in $\mathcal{E}$. Given a geometric morphism $(f_*, f^*)\colon \mathcal{F} \to \mathcal{E}$, there is an induced functor $\mathcal{E}(1, \Omega_{\mathcal{E}}) \to \mathcal{F}(1, \Omega_{\mathcal{F}})$ given by sending a subobject $e \subseteq 1$ to the subobject $f^*(e) \subseteq f^*(1) = 1$. Thus we get a map $\mathsf{Topos}^{\mathrm{op}} \to \mathsf{Cat}$; this will serve as the sentences.

Finally, given any point $p\colon \mathsf{Set} \to \mathcal{E}$ and truth value $e \subseteq 1$ in $\mathcal{E}$, we have a subset $p^*(e) \subseteq \{1\}$ in $\mathsf{Set}$, and we write $p \vDash e$ iff $p^*(e) = \{1\}$.