String diagrams for regular logic

David I. Spivak (joint with Brendan Fong)

Presented on 2018/10/27

Octoberfest

Outline

1. **Introduction**
   - Application: playing with logic
   - Implications for string diagrams
   - String diagrams for regular logic

2. Regular categories and regular logic

3. Bringing it all together

Minority Report

The 2002 movie *Minority report* showed detective Tom Cruise playing seamlessly with logic.

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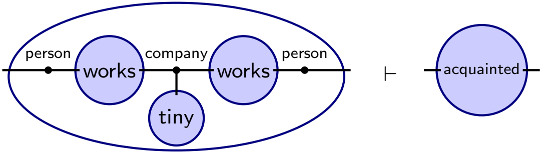
Let's imagine such a detective scenario. The knowledge base says:

- Any two people who work in the same tiny company are acquainted.
- *Categorical Informatics* is a tiny company.
- David works at *Categorical Informatics*.
- Ryan works at *Categorical Informatics*.

We of course want to conclude that David and Ryan are acquainted.

Sample scenario

Assume:



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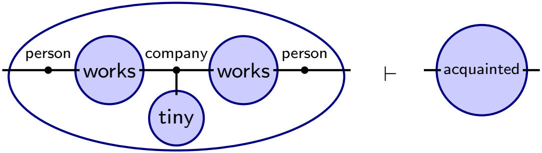

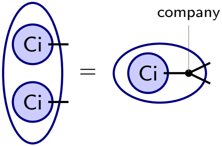

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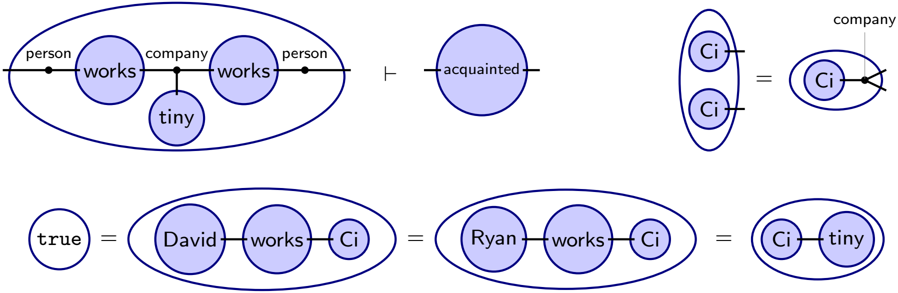

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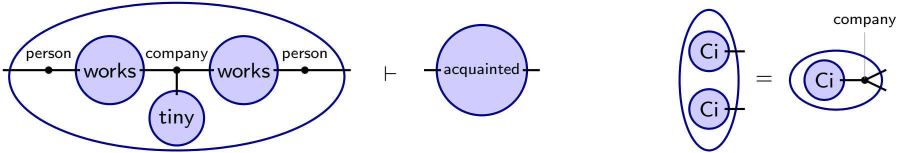
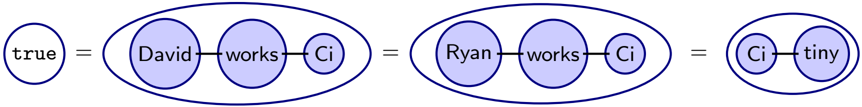
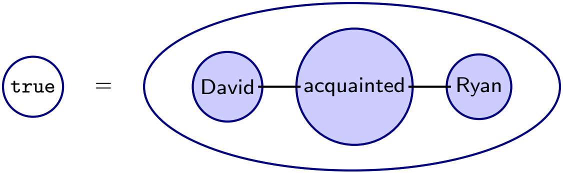
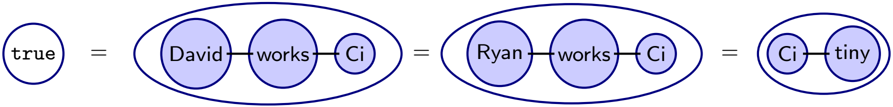




Show:



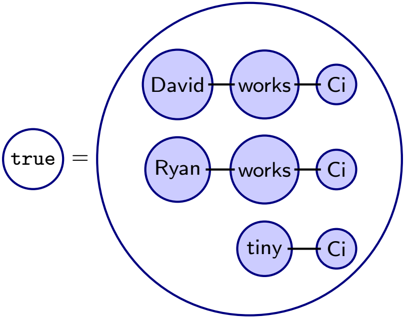
Picture proof



Combine!

Picture proof

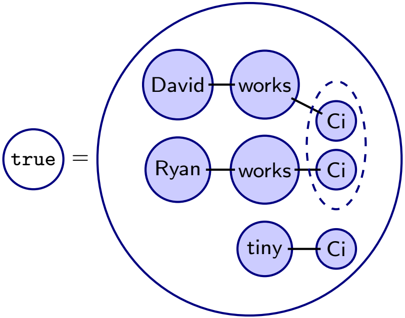
Combined:



Group two Ci's!

Picture proof

Ci's grouped:



Substitute!

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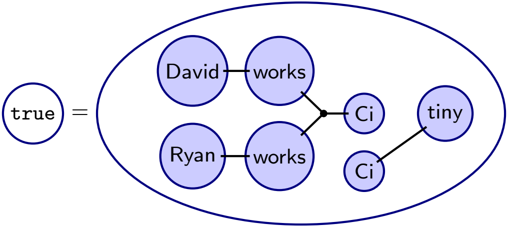
Substituted:



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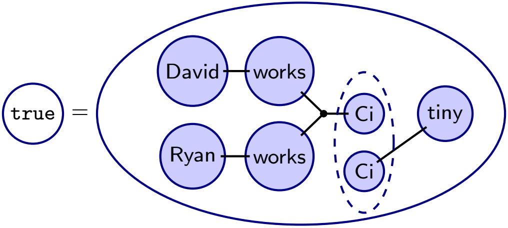

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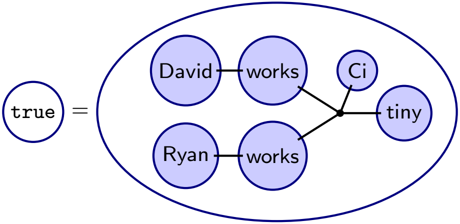

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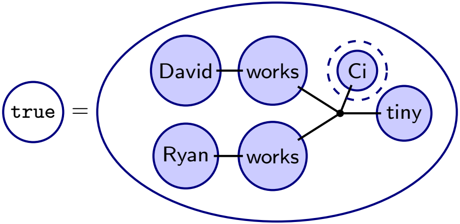
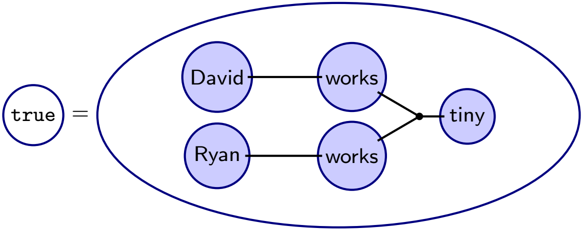


Discard group!

Picture proof

Group discarded:



Group!

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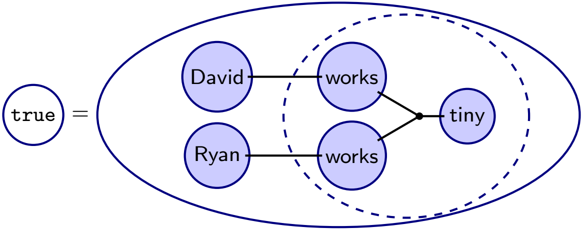
Grouped:



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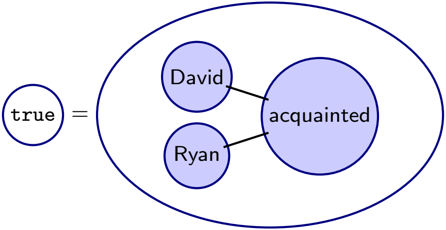
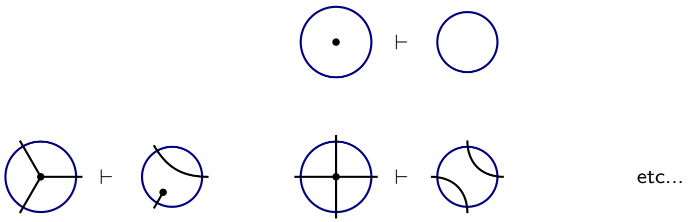


Done!

Two-dimensional manipulation of string diagrams

In this talk we discuss a 2-dimensional language for wiring diagrams.

- It includes all the sorts of operations shown above.
- Together with operations like discarding and breaking wires:



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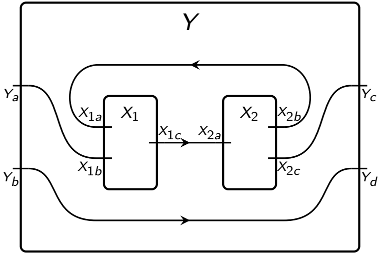

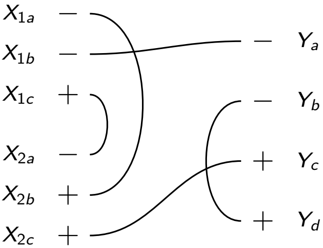

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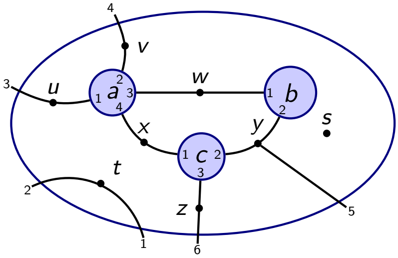
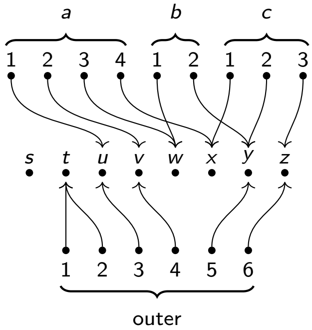
- In hypergraph categories, add Frobenius maps, plus axioms.
  - Hypergraph categories are algebras on the operad Cospan.





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- In our picture proof, we had more operations and relations.
  - Order on elements of each arity, preserved by substitution.
  - Meet-semilattice structures on elements of each arity.
  - Top element (`true`) can be discarded; corresponding structure for $\wedge$.
  - Removing dots, breaking wires.

We will see that this is a 2-dimensional structure.

Formal presentation of the calculus I.

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  - 1-morphisms: $\begin{tikzcd} n_1 \arrow[r] \arrow[dr, "v_1"'] & n_{12} \arrow[d] & n_2 \arrow[l] \arrow[dl, "v_2"] \\ & \Lambda & \end{tikzcd}$
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- Consider the (locally posetal) monoidal bicategory $\mathcal{P}\mathrm{oset}$.
  - Obj: posets; 1-morphisms: monotone maps; 2-morphisms: nat. trans.
  - Monoidal structure: $(1, \times)$.

Formal presentation of the calculus II.

We have monoidal bicategories $\mathcal{C}\mathsf{ospan}$ and $\mathcal{P}\mathsf{oset}$.

Definition

A *regular hypergraph category* is a lax monoidal 2-functor

$$T\colon \mathcal{C}\mathsf{ospan}_{\wedge}^{\mathsf{co}} \to \mathcal{P}\mathsf{oset}$$

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Throughout this talk, I'll use a new notation for adjunctions.

- Usual notation: $C \underset{L}{\overset{R}{\underset{\leftarrow}{\overset{\rightarrow}{\top}}}} D \qquad C \underset{R}{\overset{L}{\underset{\rightarrow}{\overset{\leftarrow}{\bot}}}} D \qquad D \underset{R}{\overset{L}{\underset{\leftarrow}{\overset{\rightarrow}{\bot}}}} C.$
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Regular hypergraph categories and regular categories

Denote by $\mathcal{C}$ospan-Alg the category of regular hypergraph categories, i.e. sets Λ and ajax 2-functors

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Theorem

There is an adjunction

$$\mathcal{C}\mathrm{ospan}\text{-}\mathrm{Alg} \underset{\Psi}{\overset{\Phi}{\underset{\longleftarrow}{\overset{\longrightarrow}{\Rightarrow}}}} \mathrm{RegCat}\ ,$$

such that for any regular category $\mathcal{R}$, the counit $\Psi(\Phi(\mathcal{R})) \to \mathcal{R}$ is an equivalence of categories.

Plan

- We'll return to the theorem shortly.
- First we want to recall the definition of regular categories.
- We also want to make the connection to regular logic.
    - The $\mathcal{C}$ospan-algebra story is a graphical representation of the logic.
    - This will be evident, but one can take the theorem as justification.
- Then we'll unpack the theorem and conclude.

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Regular categories

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Examples of regular categories:
- Set, and more generally any topos;
- $\mathrm{Set}^{\mathrm{op}}$, opposite of any topos, $\mathrm{TopSp}^{\mathrm{op}}$;
- The category of models of any Lawvere theory (Groups, Rings, ...);
- The slice (also the coslice) of any regular category over any object;
- Exponential ideal: if $\mathcal{R}$ regular and $\mathcal{C}$ a category, then $\mathcal{R}^{\mathcal{C}}$ is regular.

How to think of regular categories

Regular categories are those with a good *bicategory of relations*.

- A relation in $\mathcal{R}$ is a subobject $S \subseteq A \times B$.
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Regular categories have enough structure to do regular logic.

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- $\Lambda = \{A, B\}$, one relation symbol: $\vdash_{a:A,b:B} f(a,b) : \mathit{Prop}$
- Axioms:

$$
\begin{array}{ll}
f \text{ is "total":} & \mathtt{true} \vdash_{a:A} \exists(b:B).\, f(a,b) \\
f \text{ is "deterministic":} & \exists(a:A).\, f(a,b) = f(a,b') \vdash_{b,b':B} b = b'
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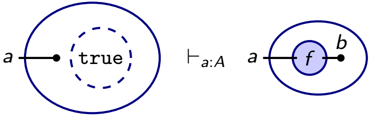
Regular logic and cospan-algebras



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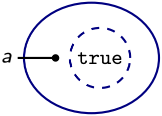

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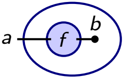

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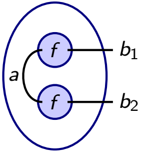

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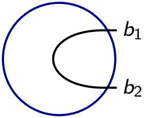

$$\exists(a : A).\, f(a, b_1) = f(a, b_2) \vdash_{b_1, b_2 : B} b_1 = b_2$$

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- Easier to see posets and adjunctions in RegCat: subobject lattices.

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One can form the *syntactic category* $\mathcal{R}_T$ of $T\colon \mathcal{C}\mathrm{ospan}_\Lambda \to \mathcal{P}\mathrm{oset}$.

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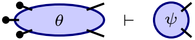

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- Dropping the ajax condition may give something like quantaloids.
- Landing in categories other than $\mathcal{P}\mathrm{oset}$ gives "fuzzy regcats."
  - E.g. $\mathcal{C}\mathrm{ospan} \to \mathcal{L}\mathrm{awvMetSp}$: "distance to entailment" $\varphi \vdash^{17} \psi$.
  - Other quantales (e.g. powerset of a monoid) give other fuzz.

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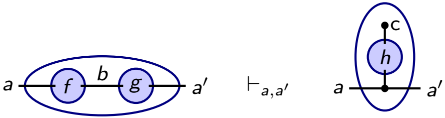

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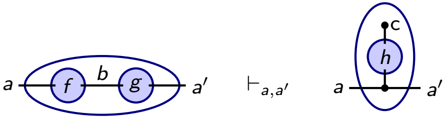

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- We have 2-reflectivity, suggesting that the diagram language is robust.

Thanks! Comments and questions welcome.