Lenses: applications and generalizations

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Outline

1 **Introduction**
- An agent in an environment
- Lenses organize interactions
- Lenses in CT

2 Some applications of lenses

3 Generalizing lens categories

4 Conclusion

An agent in an environment

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- What does that mean?
- It means agent and environment are communicating somehow.
  - The agent *observes* the environment and *acts* on it.
  - The agent's state affects that of the environment and vice versa.
    - Agent affects environment through *action*.
    - Environment affects agent through *observation*.
    - Each is affected in that it undergoes a change of *state*.

How shall we model this mathematically?

A formalization of agent/environment interaction

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- a set A for the possible actions, and
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- Environment's state is updated by the action via $S_{\text{En}} \times A \to S_{\text{En}}$.

How to organize all this stuff?

We have sets $S_{\mathsf{Ag}}, S_{\mathsf{En}}, A, O$ and functions

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- Each pair of functions is a special case of what are called *lenses*.
- Lenses are the morphisms in a cat **Lens**, whose objects are pairs $\binom{X}{Y}$.
  - The lenses from our agent/environment setup would be denoted:
  - $\binom{S_{\text{Ag}}}{S_{\text{Ag}}} \to \binom{A}{O}$ and $\binom{S_{\text{En}}}{S_{\text{En}}} \to \binom{O}{A}$

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Lenses have been coming up in the ACT community a lot lately.

Applications of lenses

There have been many uses of lens-like things over the years.

- Bidirectional transformations (Oles),
- dialectica categories and linear logic (de Paiva),
- the view-update problem in databases (Hoffman, Pierce),
- functional programming (Gibbons, Oliveira, Palmer, Kmett),
- wiring diagrams, discrete and continuous dynamical systems (Spivak),
- open economic games (Ghani, Hedges),
- supervised learning (Fong, Spivak, Tuyéras).

I'll explain a few of these as we go.

Bringing lenses into the fold

The formula lenses and their composition is kinda weird:

$$\mathbf{Lens}\left(\begin{pmatrix}A\\A'\end{pmatrix},\begin{pmatrix}B\\B'\end{pmatrix}\right) := \left\{ (f, f^\sharp) \;\middle|\; \begin{array}{r} f : A \to B \\ f^\sharp : A \times B' \to A' \end{array} \right\}.$$

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- Today: we'll first see **Lens** as part of a larger category that
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  - provides a geometrical perspective,
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  - has better formal properties.
- We then generalize further to pick up some close cousins of lenses.

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- Kmett, Riley, etc. have generalized lenses to *optics*.
  - Briefly: for any monoidal category $(\mathcal{C}, I, \otimes)$, ...
  - an optic $\binom{A}{A'} \to \binom{B}{B'}$ can be identified with an element of

$$\int^{M \in \mathcal{C}} C(A, M \otimes B) \times C(M \otimes B', A').$$

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- This can be generalized even further using Tambara modules.
- However, it's not the direction I want to go today.

Plan of the talk

Plan for the rest of the talk:

- Some applications of lenses
- Generalizing lens categories

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- Idea: if we multiply these lenses we get:

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and there's an "symmetry" lens morphism $\binom{A \times O}{O \times A} \to \binom{1}{1}$.

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We can see this as part of a bigger picture.

The agent-environment system

So what were we doing when we:

- started with lenses $\binom{S}{S} \to \binom{A}{O}$ and $\binom{S'}{S'} \to \binom{O}{A}$,
- multiplied them together to get a map $\binom{S \times S'}{S \times S'} \to \binom{A \times O}{O \times A}$, and then
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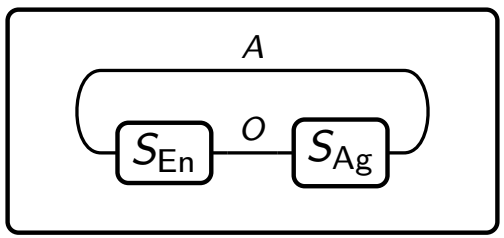

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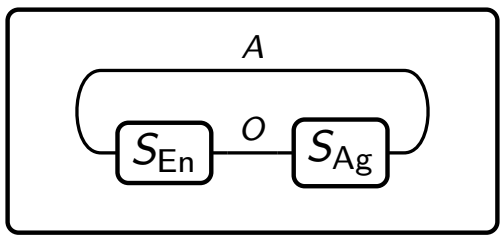
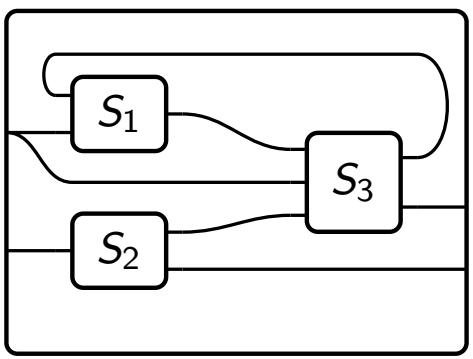


More generally we can consider open systems with many interacting agents



Wiring diagrams

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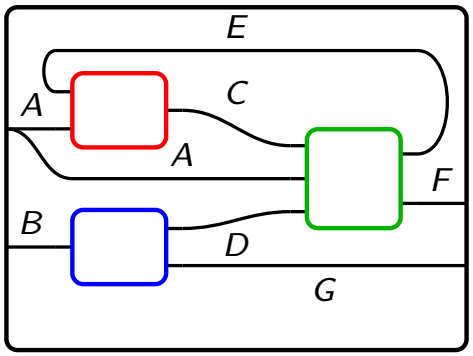

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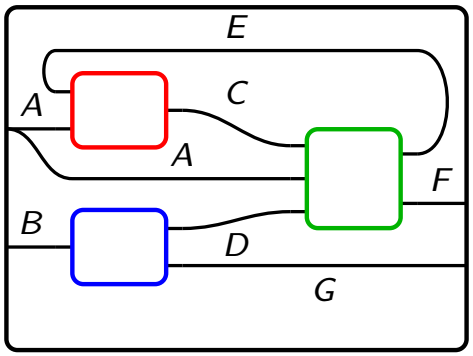

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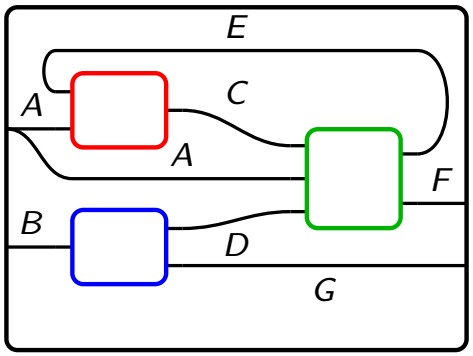

For each box, we have a pair $\binom{\text{outputs}}{\text{inputs}}$.

- We have three interior boxes: $\binom{C}{E \times A}$, $\binom{D \times G}{B}$, $\binom{E \times F}{C \times A \times D}$.
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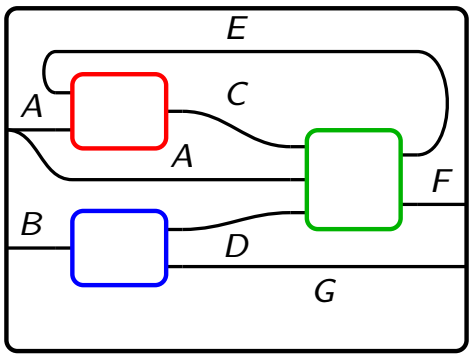

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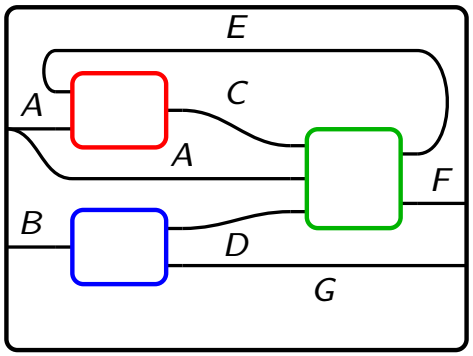

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- Both maps are just projections and diagonals:

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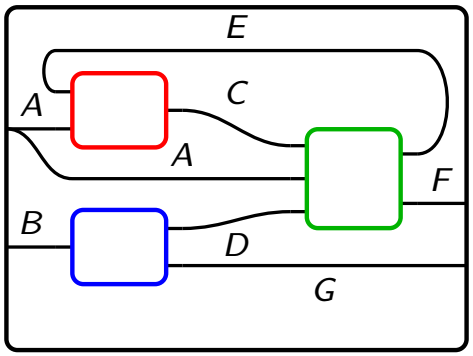

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Every wiring diagram gives a lens made of projections and diagonals.

WDs and discrete dynamical systems

A discrete dynamical system of type $\binom{A}{A'}$ consists of

- A set S
- A function $f^{\mathrm{rdt}} \colon S \to A$ called "readout"
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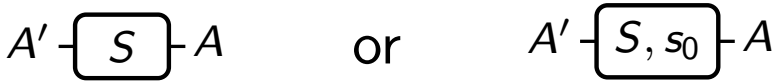

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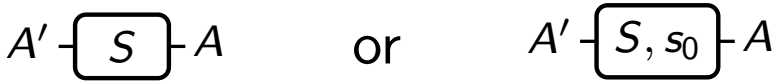

- A wiring diagram is a lens $\binom{A_1}{A_1'} \otimes \cdots \otimes \binom{A_n}{A_n'} \to \binom{B}{B'}$, and
- Each dynamical system is a lens $\binom{S_i}{S_i} \to \binom{A_i}{A_i'}$. So composing...
- We get a dynamical system $\binom{S_1 \times \cdots \times S_n}{S_1 \times \cdots \times S_n} \to \binom{B}{B'}$ in outer box.

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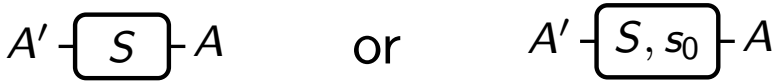

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This story of DS's and WD's existed years before I knew about lenses.

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  - It consists of a function $P \times A' \to A$, where P is a set
  - It also has an update-backprop function $P \times A' \times A \to P \times A'$.

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  - So it's just a lens $\binom{\text{implement}}{\text{upd-backprop}} : \binom{P}{P} \otimes \binom{A'}{A'} \to \binom{A}{A}$

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Learners

Similarly, the story of learners existed before we knew about lenses.

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Our category **Learn** is just **Para**(**Lens**).

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Can we do better?

Continuous dynamical systems?

Recall that a discrete dynamical system with inputs A' and outputs A is:

- A set S
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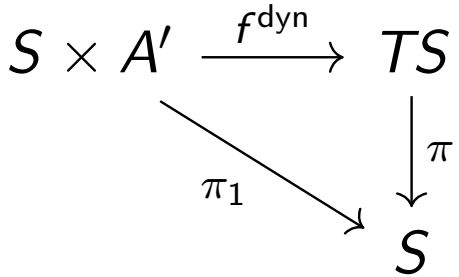

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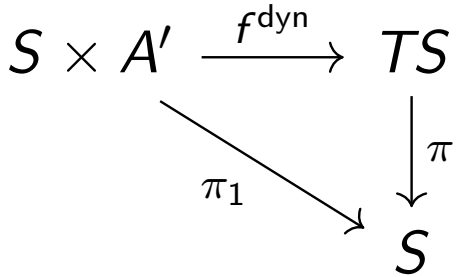

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The two notions are quite similar, but can we see the latter as a lens?

Outline

1 Introduction

2 Some applications of lenses

3 **Generalizing lens categories**

4 Conclusion

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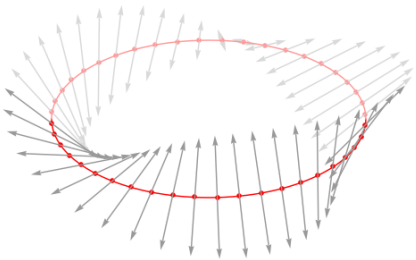
Suggestion: think of objects as "bundles."

What are bundles?



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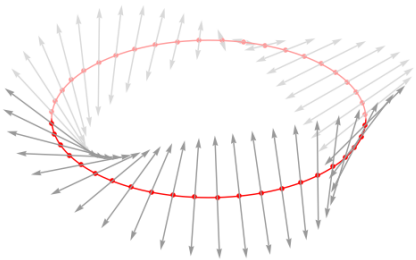

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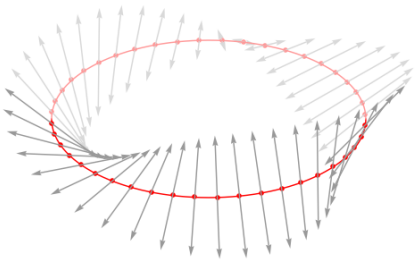

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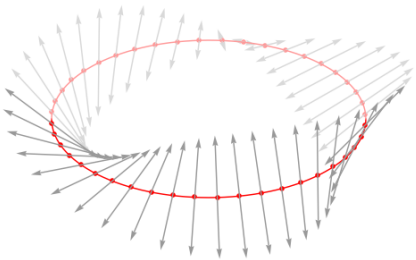

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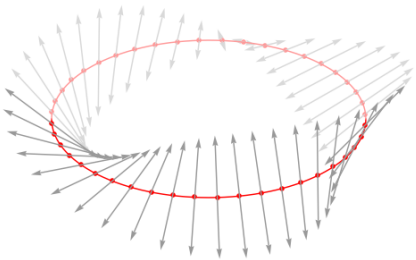

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A *trivial bundle* is one of the form $\pi_1\colon B \times B' \to B$ for some B'.

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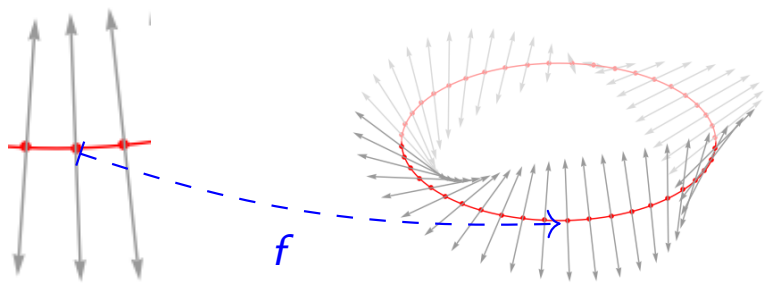

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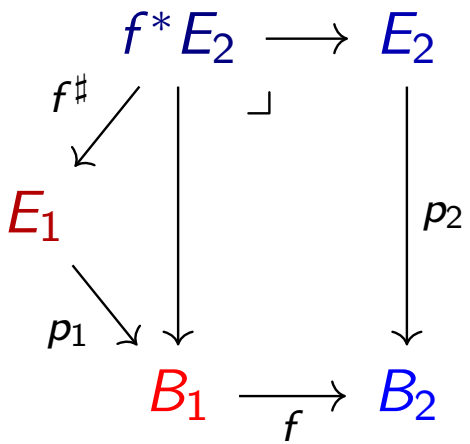

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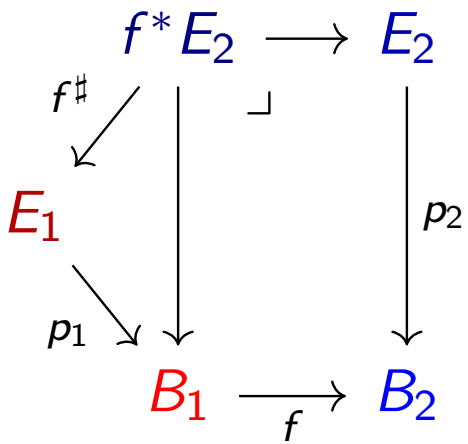

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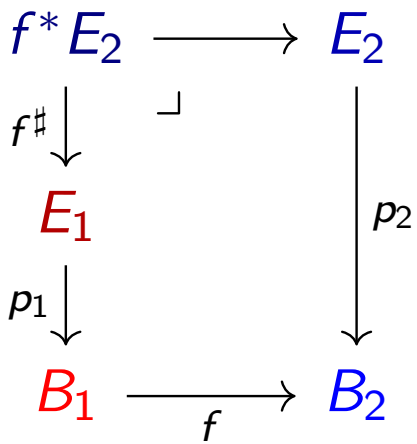

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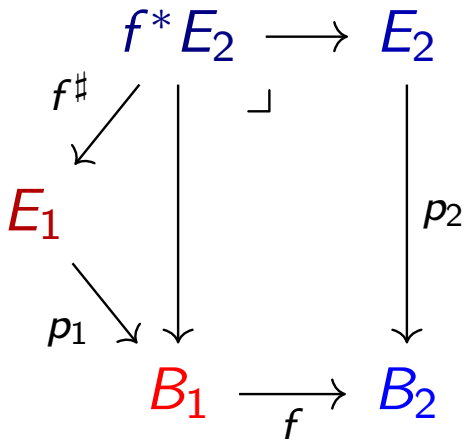

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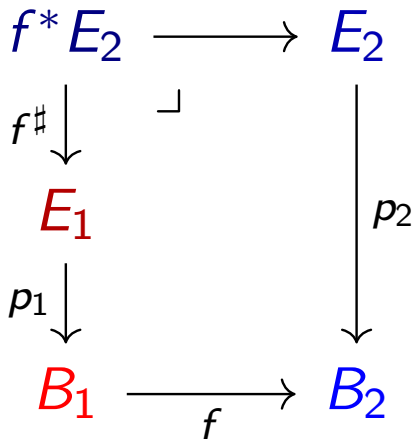

There's a strong relationship between the AG-style maps and lenses.

Example

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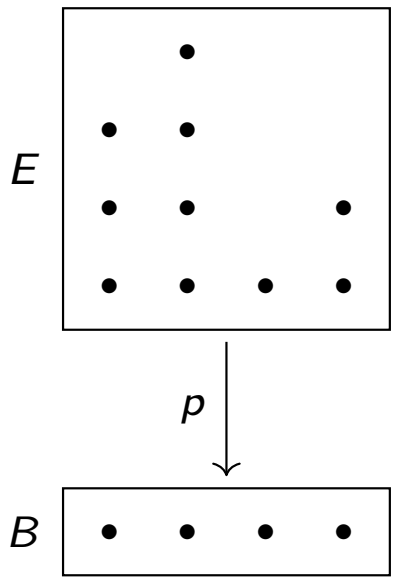

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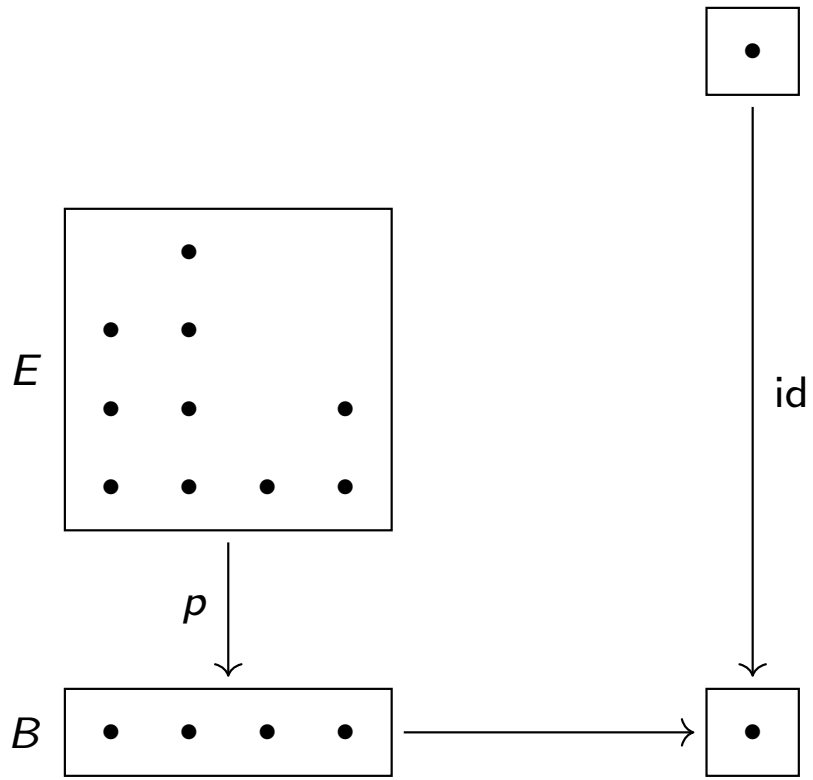

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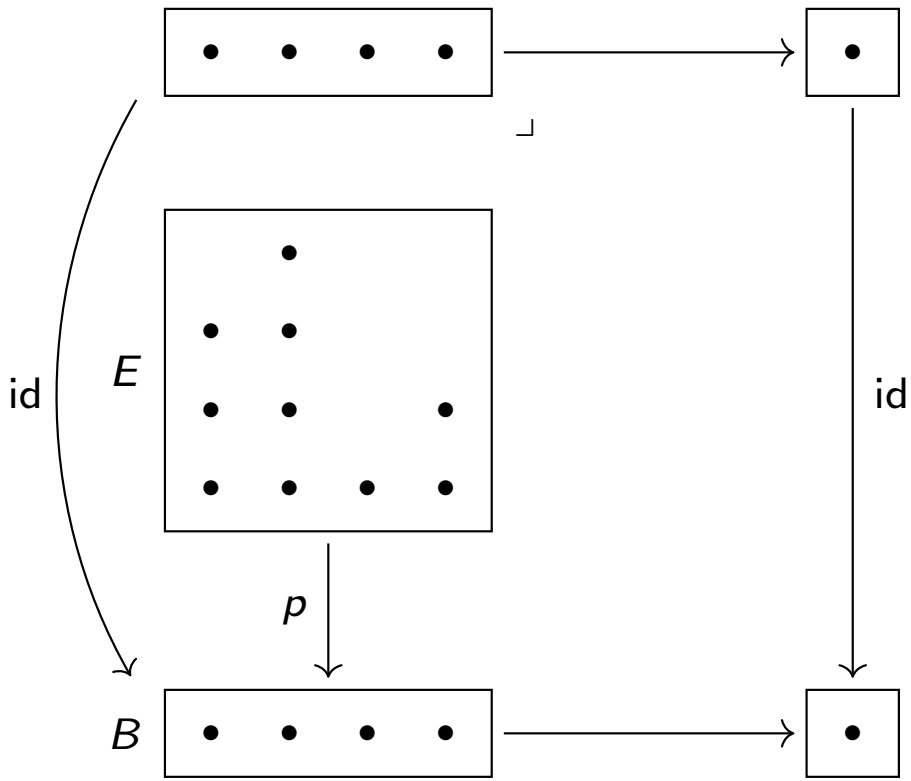

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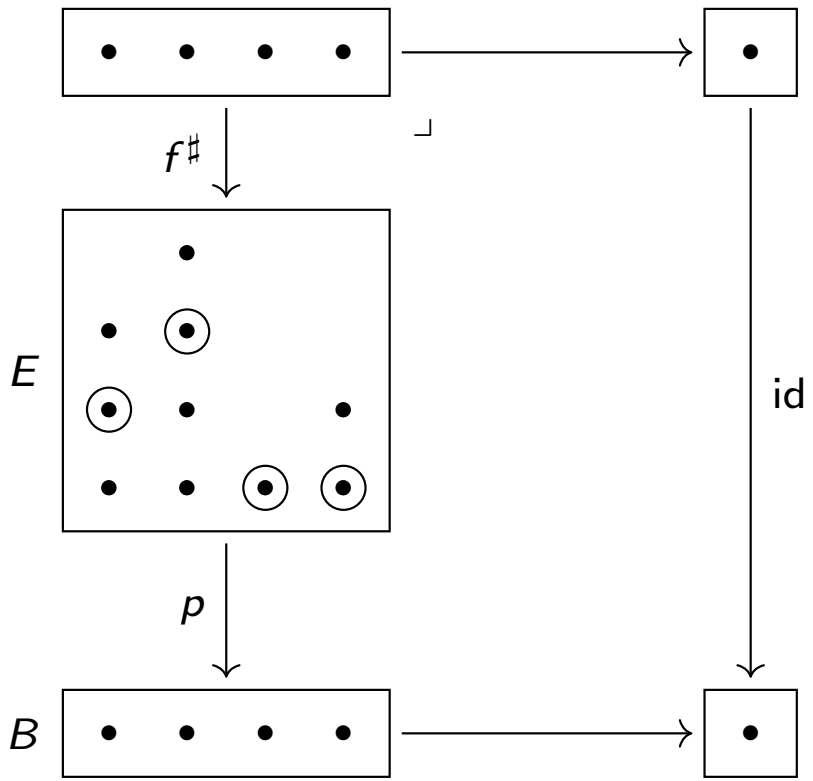

Interpretation of bimorphic lenses as trivial bundles

We will see that **Lens** sits inside this category of bundles.

- That is, there is a fully faithful functor **Lens** $\to$ **Bund**.
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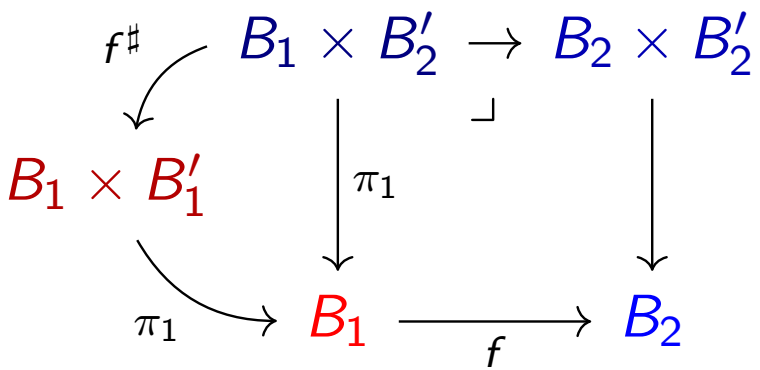

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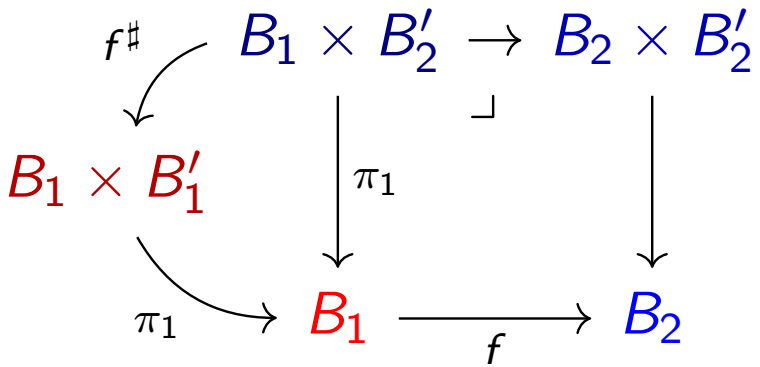

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Let's change the name **Bund** to **Lens**$'$.

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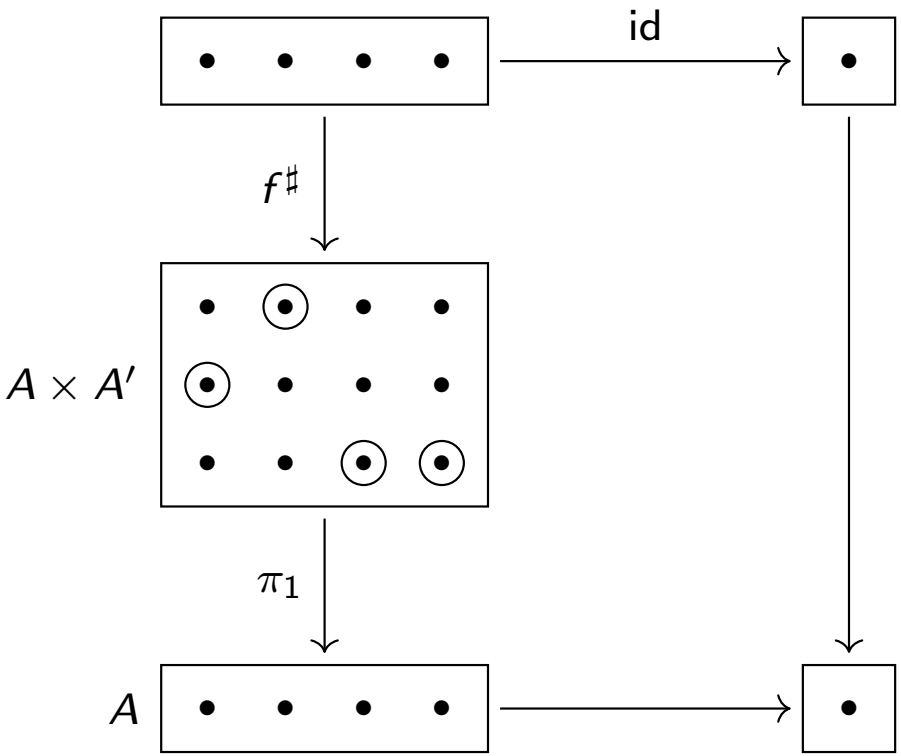

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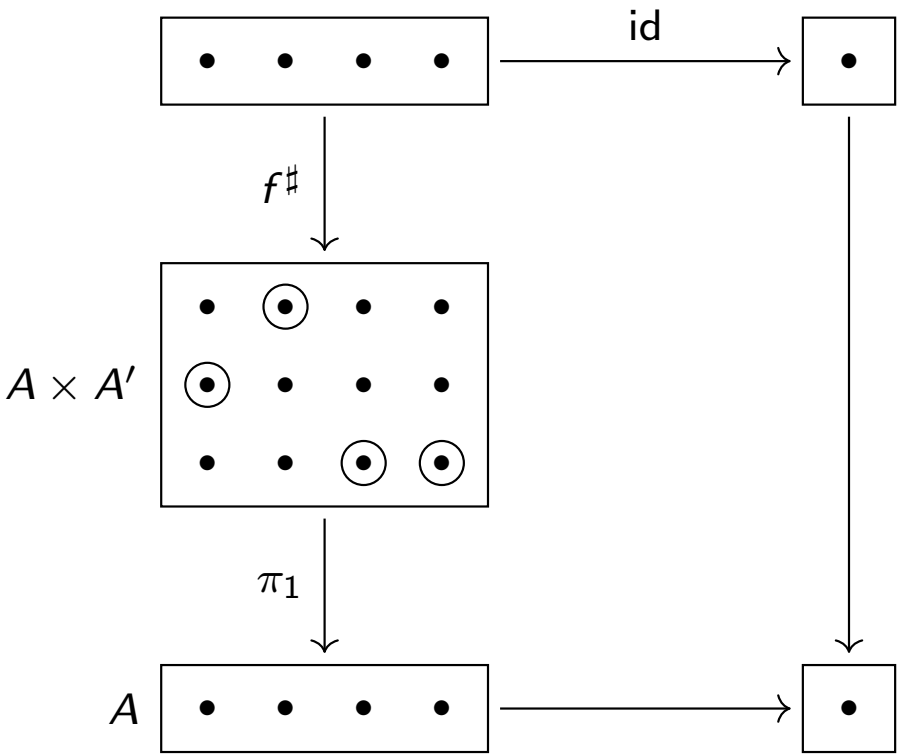

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  - Special case $A = A' = 1$, a discrete dynamical system!

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Note that $\binom{B}{B'}$ really means the trivial bundle $B \times B' \to B$.

- I've recently been denoting $\binom{B}{E}$ by $\left[\begin{smallmatrix} E \\ B \end{smallmatrix}\right]$.
- So we have $\left[\begin{smallmatrix} TS \\ S \end{smallmatrix}\right]$. Looks more bundley.
- If we want, we could reserve $\binom{A}{A'}$ for usual lenses (trivial bundles).

Ringed spaces

In algebraic geometry they study *ringed spaces* $(X, \mathcal{O}_X)$.

- Here X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on it.
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A morphism of ringed spaces $(f, f^\sharp) \colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is:

- A continuous map $f \colon X \to Y$
- A map of sheaves $f^* \mathcal{O}_Y \to \mathcal{O}_X$.
- That is, it's a map $\left[\begin{smallmatrix} \mathcal{O}_X \\ X \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{O}_Y \\ Y \end{smallmatrix}\right]$.

Continuous dynamical systems

Recall: if A', A are manifolds, a *continuous dynamical system* is:

- A manifold S, (tangent bundle TS),
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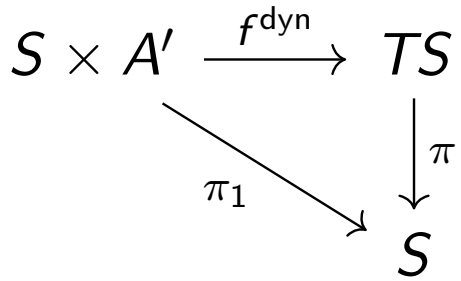

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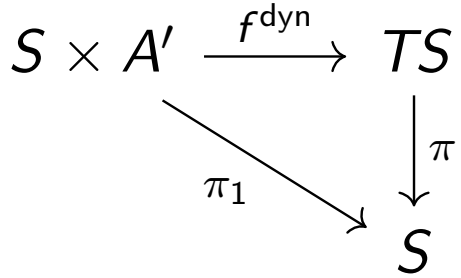
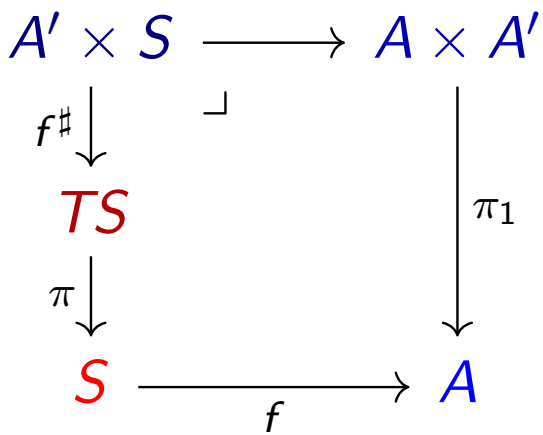


But from the bundle perspective that commutative diagram is baked in.



In other words the dynamical system is just a lens map $\left[\begin{smallmatrix} TS \\ S \end{smallmatrix}\right] \to \left[\begin{smallmatrix} A' \\ A \end{smallmatrix}\right]$

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- The following are equivalent:
  - The opposite of the covariant Grothendieck construction for $\mathcal{E}$.
  - The contravariant Grothendieck for the pointwise opposite of $\mathcal{E}$.
  - The category with
    - objects $\{(B, E) \mid B \in \mathcal{B}, E \in \mathcal{E}(B)\}$
    - morphisms $(B_1, E_1) \to (B_2, E_2)$ given by "lensy" $(f, f^\sharp)$.

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- Take $\mathcal{B} := \{\text{comonoids } (c, \epsilon, \delta) \text{ in } \mathcal{M}\}$
- Take $\mathcal{E}(c) := \mathbf{coKl}(c \otimes -)$, the coKleisli cat. of comonad $x \mapsto c \otimes x$.
- In $\left[\begin{smallmatrix} m \\ c \end{smallmatrix}\right]$, think of m as the product coalgebra $c \otimes m$, "trivial bundle".

The right generalization of Lens?

We've seen many different lens-like categories $\mathcal{L}$.

- For each, there's a category $\mathcal{B}$ and a functor $\mathcal{E} : \mathcal{B}^{\mathrm{op}} \to \mathbf{Cat}$ ...
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But this is perhaps overly general.

- Every split fibration is $\mathbf{Lens}_{\mathcal{E}}$ for some $\mathcal{E}$.
- Should we be looking for something more specific than this?
- Example: the twisted arrow category is $\mathbf{Lens}_{\_/\mathcal{C}}$; too general?

$$\begin{array}{ccc} E_1 & \xleftarrow{f^\sharp} & E_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ B_1 & \xrightarrow[f]{} & B_2 \end{array}$$

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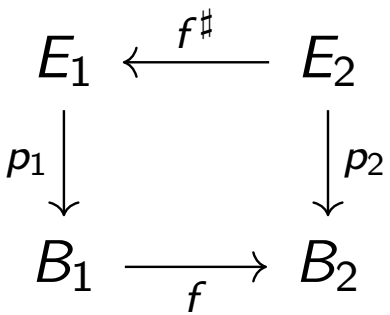

A morphism $\begin{bmatrix} E_1 \\ B_1 \end{bmatrix} \to \begin{bmatrix} E_2 \\ B_2 \end{bmatrix}$ in the twisted arrow category. Looks kinda lensy, but should it count?

Formal properties of $\mathbf{Lens}_{\mathcal{E}}$

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- Then $\mathbf{Lens}_{\mathcal{B}/-}$ has excellent formal properties.
  - Complete, cocomplete, Cartesian closed.
  - Initial alg's and final coalg's for polynomial endofunctors.
  - Another fact'n system: $\begin{bmatrix} f^\sharp \\ f \end{bmatrix}$ factors as $\begin{bmatrix} E_1 \\ B_1 \end{bmatrix} \to \begin{bmatrix} f_* E_1 \\ B_2 \end{bmatrix} \to \begin{bmatrix} E_2 \\ B_2 \end{bmatrix}$

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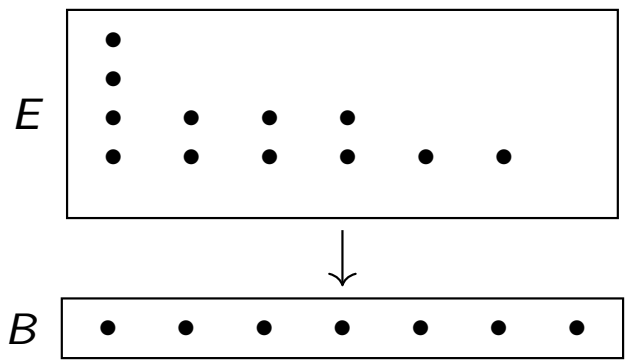

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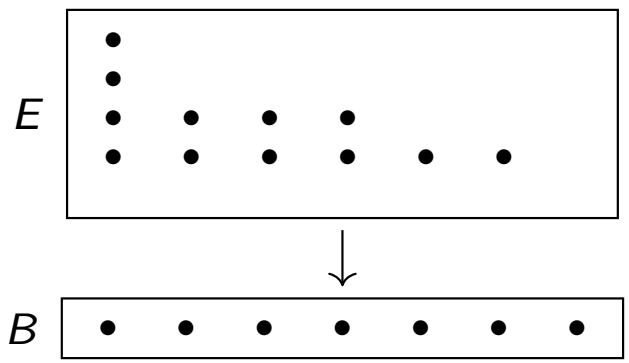

- And they have the same morphisms too: $\mathbf{Poly}_{\mathcal{B}} \cong \mathbf{Lens}_{\mathcal{B}/-}$.

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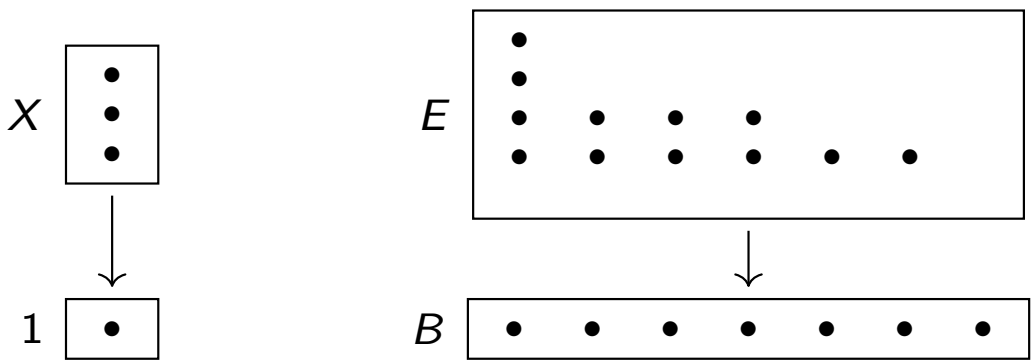

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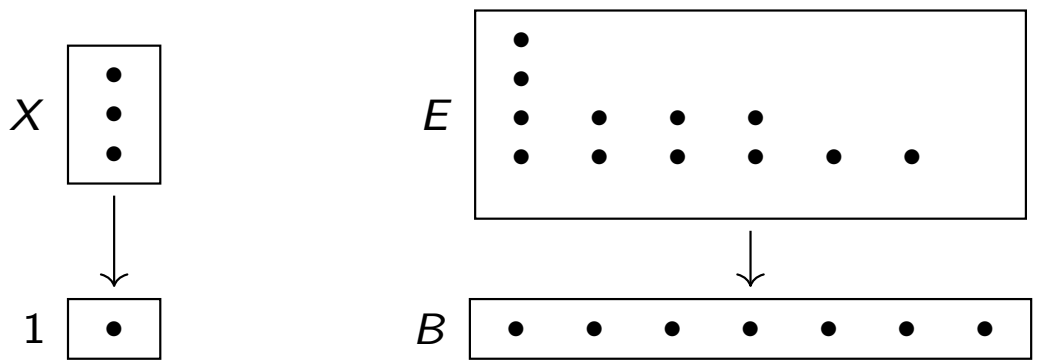

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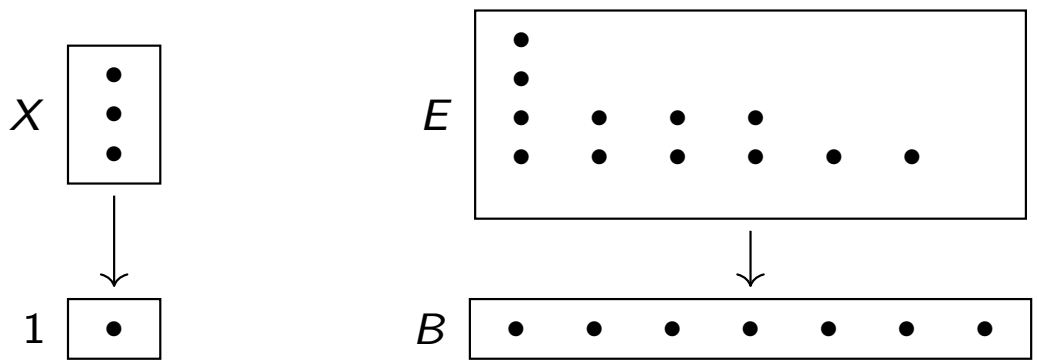

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- The functor acts on a lens $\left[\begin{smallmatrix} E \\ B \end{smallmatrix}\right] \to \left[\begin{smallmatrix} E' \\ B' \end{smallmatrix}\right]$ by composing with it.

Outline

1 Introduction

2 Some applications of lenses

3 Generalizing lens categories

4 Conclusion

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