Graphical calculus for abelian categories

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May 2, 2019

Abstract

Abelian categories are "convenient places to calculate", e.g. to do homological algebra. One nice feature of abelian categories is that they are regular, meaning that each has a well-working calculus of relations. This calculus can be given a graphical formulation, with a "user interface" of wiring diagrams that we can specify mathematically.

I'll present abelian categories from this point of view, with the following compressed mathematical specification: a graphical abelian calculus is a bi-ajax bi-functor P: LinRel–>Poset. Unpacking this, I'll explain that there is a graphical syntax of linear relations, which is a mild ("thin") extension of Sobociǹski's graphical linear algebra. It forms a bi-category LinRel, and functors from LinRel to the bi-category of posets provide semantic content to this syntax: they tell us how we can "fill the shells". Any lax monoidal bi-functor P: LinRel–>Poset whose laxators have both left and right adjoints gives rise to an abelian category, and all abelian categories (up to equivalence) arise in this way.

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I. Introduction

A. Abelian categories

1. Examples — 1 min
   a. Vector spaces (over a fixed field k, or letting k vary)
   b. Abelian groups
   c. Finitely generated abelian groups
   d. Sheaves of abelian groups on a space
2. As good places to do computation — 1 min
   a. Homological algebra
   b. Roughly: think column spaces, nullspaces of matrices
3. Definition: category with — 2 mins
   a. A zero object
   b. Every pair of objects has a product and a coproduct
   c. Every morphism has a kernel and cokernel
   d. Every monic is a kernel, every epic is a cokernel

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B. Graphical language — 3 mins

1. Draw: $\{(a_1, a_2) \in A \times A \mid \exists a', b', c'.\, R(a_1, b', c') \wedge S(a', b') \wedge a_2 = a_1 + 2a'\}$.
2. The logic and pictures take place in a bi-category called $\mathbb{L}\mathsf{inRel}$.

C. Theorem

1. An abelian calculus is a pair (T, P) where — 1 min
   a. T is a set
   b. $P\colon \mathbb{L}\mathsf{inRel}_{\mathrm{T}} \to \mathbb{P}\mathsf{oset}$ is a bi-ajax bi-functor
2. Adjunction $\mathbf{syn}\colon \mathsf{AbCalc} \leftrightarrows \mathsf{AbCat} : \mathbf{prd}$ — 3 mins
   a. For abelian $\mathscr{A}$, have equivalence $\mathbf{syn}(\mathbf{prd}(\mathscr{A})) \simeq \mathscr{A}$.
3. Different knobs to turn — 2 mins
   a. Rather than four axioms on category
   b. Specific structures that can be varied.

D. Plan: — 2 mins

1. Abelian categories

2. Graphical language: abelian calculi
3. Relationship between them

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II. Properties of abelian categories

A. Let $\mathscr{A}$ be abelian. Then — 2 mins

1. $\mathscr{A}^{\mathrm{op}}$ is abelian.
2. For any small category I, the functor category $\mathscr{A}^I$ is abelian.
3. $\mathscr{A}$ is balanced: if f is both epic and monic, then it is an iso.
4. $\mathscr{A}$ has all finite limits and colimits.
5. $\mathscr{A}$ has epi-mono factorization system.
6. $\mathscr{A}$ is enriched in Ab.
7. Biproducts: for all $A, B \in \mathscr{A}$, the natural map $\langle [A, O], [O, B] \rangle \colon A \oplus B \to A \times B$ is an isomorphism.
8. For any $A \in \mathscr{A}$, the poset $\mathsf{Sub}(A)$ has finite meets and joins.
9. The pushout of a monomorphism along any map is again a monomorphism.

B. $\mathscr{A}$ is regular

1. Usual def: — 2 mins
   a. Has finite limits
   b. Has image factorizations
   c. Image factorizations are pullback-stable
2. For us: it has a bi-category $\mathbb{R}\mathsf{el}_{\mathscr{A}}$ of relations. — 3 mins
   a. $\mathrm{Ob}\,\mathbb{R}\mathsf{el}_{\mathscr{A}} = \mathrm{Ob}\,\mathscr{A}$, $\mathbb{R}\mathsf{el}_{\mathscr{A}}(X, Y) = \{R \subseteq X \times Y\}$
   b. Hom-posets have finite meets.
   c. $\mathbb{R}\mathsf{el}_{\mathscr{A}}$ supports a graphical calculus
      (1) We've discussed it before (pic)
      (2) It's part of a richer graphical calculus which we'll discuss today.
3. Recover $\mathscr{A}$ as the category of left adjoints in $\mathbb{R}\mathsf{el}_{\mathscr{A}}$.

C. $\mathscr{A}$ has more structure: — 4 mins

1. Regular cats have a logic with $\exists, =, \mathtt{true}, \wedge$.
   a. Example: composing relations $\{(a, c) \mid \exists b.\, R(a, b) \wedge S(b, c)\}$.
2. Abelian cats extend that logic with $0, +$.

3. $\{(a_1, a_2) \in A \times A \in | \exists a' \in A.\, a_1 + 3a_2 = a'\}$.
4. This allows us to take joins of subspaces R, S: $\{x \mid \exists x_1, x_2.\, R(x_1) \wedge S(x_2) \wedge x = x_1 + x_2\}$.

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III. Abelian calculi $\mathbb{L}\mathsf{inRel}_{\mathrm{T}} \to \mathbb{P}\mathsf{oset}$

A. Goal — 2 mins

1. Encode the additional structure in a syntax category
2. Get totally different take on abelian cats.

B. $\mathbb{L}\mathsf{inRel}$

1. Lin: the category of integer matrices — 3 mins
   a. $\mathrm{Ob}\,\mathsf{Lin} = \mathbb{N}$
   b. $\mathsf{Lin}(m, n) = \{M : \underline{m} \times \underline{n} \to \mathbb{Z}\}$
   c. Full subcat of Ab spanned by $\mathbb{Z}^n$'s
   d. It's regular.
2. $\mathbb{L}\mathsf{inRel}$ is the bi-category of relations in it — 1 min
3. We can draw objects and morphisms in $\mathbb{L}\mathsf{inRel}$ — 5 mins
   a. Each object $n \in \mathbb{L}\mathsf{inRel}$ is a shell with n ports
   b. Monoidal product: trenchcoat
   c. Operadic morphisms: wiring diagrams with dots and boxes

$$\{(x, y, z) \mid x + y - 2z = 0 \wedge 2x + 2y = 0\}$$

   d. Composition: nesting of WDs

C. $P : \mathbb{L}\mathsf{inRel} \to \mathbb{P}\mathsf{oset}$ bi-ajax bi-functor

1. Drawings — 3 mins
   a. To each shell $\Gamma \in \mathbb{L}\mathsf{inRel}$, a poset $P(\Gamma)$ of fillers ("subspaces")
   b. A way of consistently combining fillers for any dot-and-box diagram

$$\{z \mid R(x, y) \wedge S(y) \wedge x + y - 2z = 0 \wedge 2x + 2y = 0\}$$

2. bi-functor: combining fillers is monotonic — 2 mins
3. bi-ajax — 2 mins
   a. lax monoidal $\boxplus : P(m) \times P(n) \to P(m + n)$ and $0 : 1 \to P(0)$.
   b. laxator has both a left and a right adjoint

D. $\mathbb{L}\mathsf{inRel}_{\mathrm{T}} = \coprod_{t \in \mathrm{T}} \mathbb{L}\mathsf{inRel}$; “strings have labels in T”.

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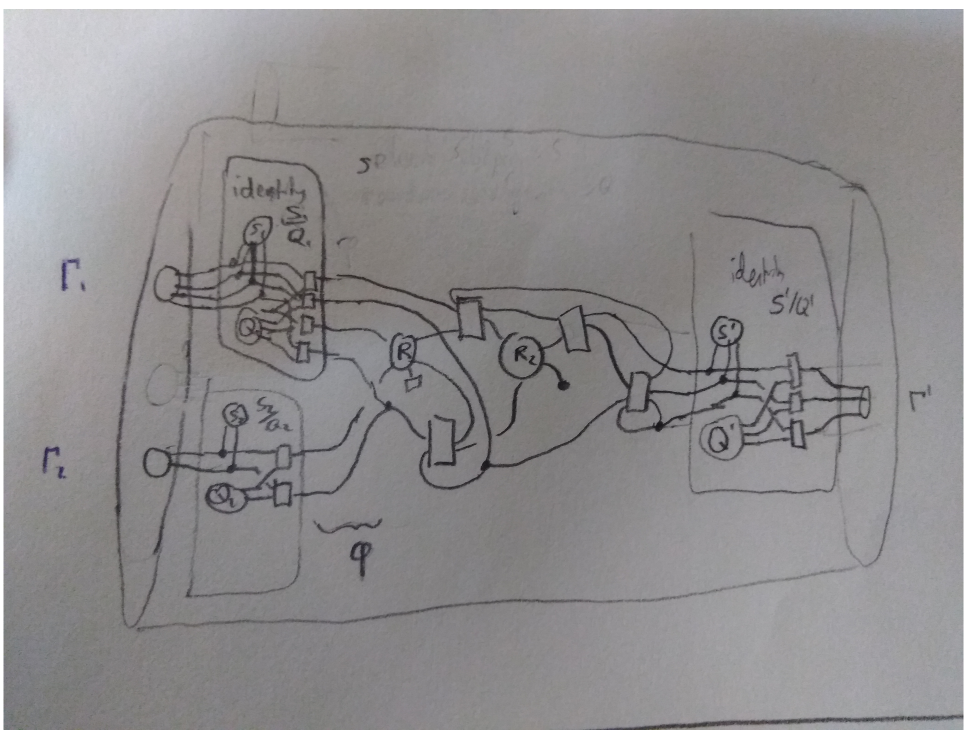
IV. The $(\mathbf{syn}, \mathbf{prd})$ adjunction

A. Given an abelian category $\mathscr{A}$, form $\mathbf{prd}(\mathscr{A})\colon \mathbb{L}\mathsf{inRel}_{\mathrm{Ob}\,\mathscr{A}} \to \mathbb{P}\mathsf{oset}$

1. On $\Gamma = (\tau\colon \underline{n} \to \mathrm{T})$, let $\mathbf{prd}(\Gamma) := \mathsf{Sub}(\prod_{i \in \underline{n}} \tau_i)$ — 1 min
2. Same on objects and morphisms as in any regular category. — 1 min
3. Bi-ajax: — 2 mins
   a. Laxator $\boxplus\colon \mathsf{Sub}(\Gamma) \times \mathsf{Sub}(\Gamma') \to \mathsf{Sub}(\Gamma + \Gamma')$
   b. Left-adjoint: project subspaces of biproduct onto the two factors
   c. Right adjoint: intersect subspaces of biproduct with the two spanning inclusions

B. Given bi-ajax $P\colon \mathbb{L}\mathsf{inRel}_{\mathrm{T}} \to \mathbb{P}\mathsf{oset}$ form $\mathbf{syn}(P)$ as follows:

1. $\mathrm{Ob}\,\mathbf{syn}(P) := \{(\Gamma, q, s) \mid q \vdash s \text{ in } P(\Gamma)\}$. — 1 min
   a. Idea: subquotients s/q.
2. $\mathbf{syn}(P)((\Gamma_1, q_1, s_1), (\Gamma_2, q_2, s_2)) := \{\theta \in P(\Gamma_1 + \Gamma_2) \mid q_1 \boxplus q_2 \vdash \theta \vdash s_1 \boxplus s_2,\ \theta \text{ is an adjunction}\}$ — 2 mins
3. Identity on (Γ, q, s) — 3 mins
   a. Quotient: blurring
   b. Subobject: filtering



C. Adjunction $\mathbf{syn}\colon \mathsf{AbCalc} \leftrightarrows \mathsf{AbCat} : \mathbf{prd}$ 1 min

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V. Conclusion 3 mins

A. Abelian cats are cool, and axioms are harmonious

B. Another perspective: functors from linear relations to posets.

C. Graphical linear algebra is math spec of UI.

D. Different knobs to turn.

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