

Categorical Databases

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Outline

An invitation to engage with us, and solve real-world problems.

1 Introduction

- The fabric of interdisciplinarity
- Our historical moment
- Plan of the talk

2 The problem

3 The math

4 The tool

5 Conclusion

A road to true interdisciplinarity

- Scientific disciplines are conceptual analogies of the world.
 - Science: a schematic, conceptual account of phenomena.
 - Engineering is using these accounts to channel world events.
 - But how do different disciplines and accounts cohere?
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- We need a shared fabric, a substrate for interdisciplinarity.
 - Interdisciplinarity consists of effective analogy-making.
 - To go further, we need to formalize the analogies themselves.
- Better yet: we need a conceptual stem-cell.
 - Something that can differentiate into huge variety of forms.
 - Find the analogies between forms as aspects within the stem cell.

Category theory as conceptual stem-cell

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- All forms of pure math... (we'll briefly discuss this)
- Databases and knowledge representation (categories and functors)
- Functional programming languages (cartesian closed categories)
- Dynamical systems and fractals (operad-algebras, co-algebras)
- Shannon entropy (operad of simplices)
- Hierarchical planning generalized lenses
- Partially-ordered sets and metric spaces (enriched categories)
- Higher order logic (toposes = categories of sheaves)
- Measurements of diversity in populations (magnitude of categories)
- Collaborative design (enriched categories and profunctors)
- Petri nets and chemical reaction networks (monoidal categories)
- Quantum processes and NLP (compact closed categories)

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 - CT—like math—explains, models, formalizes many many things.
 - Conclude that math/CT explains everything and hence nothing?
- Stem cells don't do work until they differentiate.
 - “Adult-level” work requires differentiation and optimization.
 - But the unified origins lead to impressive interoperability.

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- Designed to transport theorems from one area of math to another.
 - From topology (shapes) to algebra (equations).
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- And it's branched out from math in a big way.
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Our historical moment

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If you care about *information hygiene*, CT needs to be on your radar.

Getting to specifics

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- What is “model-space”?

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- The problem: multiple models of similar information
- What is “model-space”?
- Category theory offers a mathematical notion of model-space.
- The kinematics of data: how it moves and rests.

Plan of the talk

- The problem: pervasive and insidious.
- The math: Category theory describes kinematics of data.
- The tool: Open-source implementation and commercialization.

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2 The problem

- The Copernican revolution continues
- Information kinematics

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- Linear Algebra studies coordinate systems *and transformations*.
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 - E.g. OMOP in EMRs
 - BFO, CIDOC, SUMO, etc., etc. in upper ontologies
- Let's change focus to **transformations**.

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Multiplicity of perspectives is not going away. Let's learn to integrate.

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■ Information kinematics:

- Information rests in databases.
- Information moves by **data transformations**.
- Let's dig in.

Information kinematics

Information rests primarily in databases.

- Domain knowledge informs the structure of the database.
 - The structure is called the database *schema*.
 - It consists of a collection of interlocking tables.
- The data itself is structured according to the schema.
 - Unstructured data is not yet informative.
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 - The result of a query arrives a schema with only one table.
 - This is a limitation of current notions of querying.
- Other transformations: **ETL, schema evolution, warehousing**

Think vector spaces and linear transformations.

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- Each table represents some sort of *entity*:
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 - Its columns represents aspects of that entity.
- Example: name and owner are aspects of a house-cat.
 - The house-cat is an entity.
 - The house-cat table has a name column.
 - The house-cat table has an owner column.
 - A house-cat owner is a person, an entity of type person.

House-cat	Name	Owner	Person	Name
C101	Prince Charming	P52	P17	Alice
C241	Patches	P52	P52	Bob
C468	Mittens	P81	P81	Carl

The house-cat schema

Domain knowledge:

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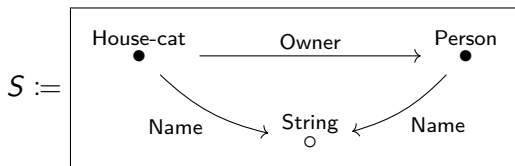
The database collects worldly examples of this knowledge:

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String
Mittens
Patches
...

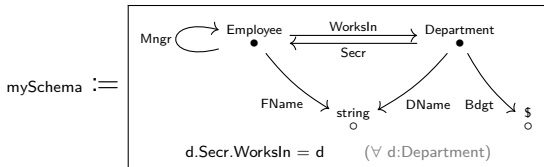
The schema for this knowledge can be drawn as a graph:



Each column connects its table to another “foreign” table.

A bit more interesting

Let's add loops and integrity constraints:



Employee	FName	WorksIn	Mngr
1	Alan	101	2
2	Ruth	101	2
3	Kris	102	3

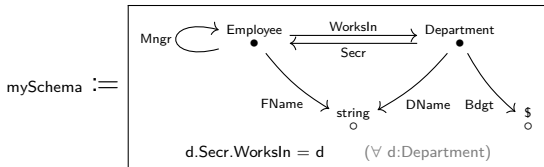
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Alan
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Stats:

1. Three dots, three tables, three ID columns.
2. Six arrows, six non-ID columns.

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- That's a **query**.

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FOR      e:Employee
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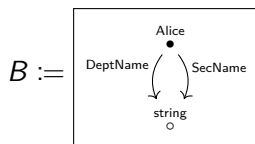
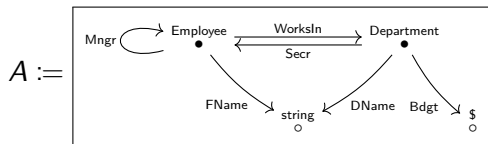
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- It's a way of transforming data: form A to form B:



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- Information management is perhaps the biggest problem today.
 - Calculus and diff. eq.? We can hire people to do that.
 - But 40% of IT budgets are spent on information integration.
 - We're constantly breaking and reviving Humpty Dumpty.
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 - IT culture doesn't even seem to name this problem explicitly.

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 - AI beyond ML requires information agility.

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3 The math

- What's a category?
- Data as set-valued functor
- Functorial schema mapping and data migration
- Data transformations
- Databases and RDF

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$$\text{src}, \text{tgt}: \text{Arr}(\mathcal{C}) \rightarrow \text{Ob}(\mathcal{C}),$$

assigning to each arrow its *source* and its *target* object, respectively.

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- A *path in* $\mathcal{C}$ is a finite “head-to-tail” sequence $f_1 \circ \cdots \circ f_n$ of arrows

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- 3 An notion of equivalence for paths, denoted $\simeq$.

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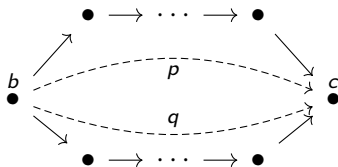
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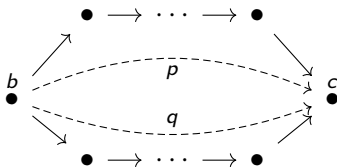
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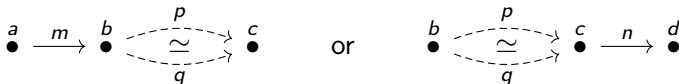
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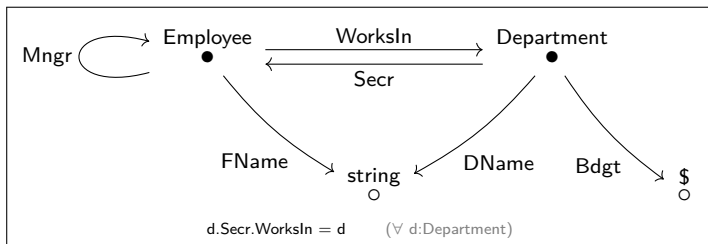


If $p \simeq q$ then for any extensions



We have equivalences: $m \circ p \simeq m \circ q$ and $p \circ n \simeq q \circ n$.

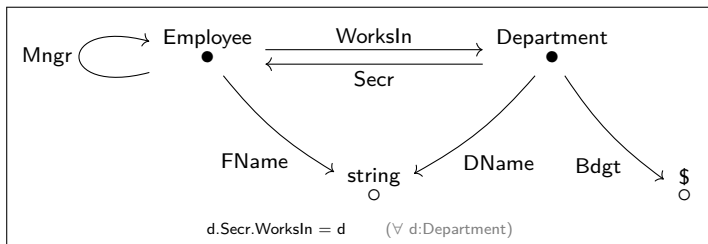
Categories = database schemas



■ Database schemas are categories!

- The objects of the category $\mathcal{C}$ are tables.
- The arrows of $\mathcal{C}$ are columns, connecting one table to another.
- The integrity constraints are path equations.
- We brush some details under the rug (distinction between $\bullet$ and $\circ$).

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- But there are also categories that are well-known in math.

Categories from mathematics

Categories are everywhere in mathematics.

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There's also a notion of *mapping* between categories: functors.

Functors: mappings between categories

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- A functor is a graph mapping that respects path equivalence.

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- A functor is a graph mapping that respects path equivalence.
- **Definition:** A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ consists of
 - a function $\text{Ob}(\mathcal{C}) \rightarrow \text{Ob}(\mathcal{D})$ and
 - a function $\text{Arr}(\mathcal{C}) \rightarrow \text{Path}(\mathcal{D})$,such that F
 - respects sources and targets,
 - respects equivalences of paths.

Functors and databases

Recall:

- A category $\mathcal{C}$ is basically a database schema.
- A functor $\mathcal{C} \rightarrow \mathcal{D}$ is a graph mapping that preserves equivalences.

Functors and databases

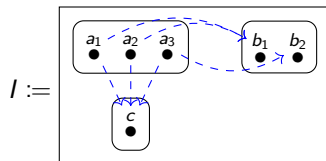
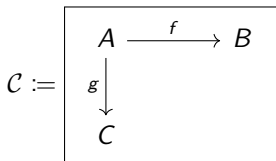
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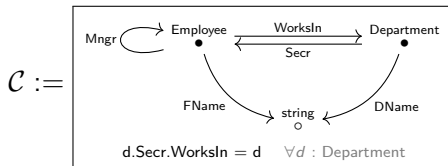
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 - two paths are equivalent if they compose to the same function.
- A functor $\mathcal{C} \rightarrow \mathbf{Set}$ fills schema $\mathcal{C}$ with data.
- Example: Let $\mathcal{C}$ be the category on the left.
- Then here's an example functor $I: \mathcal{C} \rightarrow \mathbf{Set}$:



Schema=Category, Instance=Set-valued functor

- Let $\mathcal{C}$ be the following category



- A functor $I: \mathcal{C} \rightarrow \mathbf{Set}$ consists of
 - A set for each object of $\mathcal{C}$ and
 - a function for each arrow of $\mathcal{C}$, such that
 - the declared equations hold.
- In other words, I fills the schema with compatible data.

$I :=$

Employee	FName	WorksIn	Mngr
1	Alan	101	2
2	Ruth	101	2
3	Kris	102	3

Department	DName	Secr	Bdgt
101	Sales	1	\$10
102	IT	3	\$5

Summary of the connection

- The connection between categories and databases is simple.
- A database schema is a custom category.
- Functors $I: \mathcal{C} \rightarrow \mathbf{Set}$ are database instances.
- What about functors $F: \mathcal{C} \rightarrow \mathcal{D}$ between schemas?

Data transformations

We want to move data between different frameworks.

- Data is resting in schema $\mathcal{C}$.
- We want to move it in a specific way to schema $\mathcal{D}$.
- We can specify this transformation using functors.

Functorial data migration

We can do all sorts of [data transformations](#) using functors.

- Queries, ETL processes, warehousing, schema evolution, etc.

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 - They connect schemas to schemas, $F: \mathcal{C} \rightarrow \mathcal{D}$
 - They connect schemas to data, $I: \mathcal{D} \rightarrow \mathbf{Set}$.
 - Upshot: one can compose and get a functor $(F \circ I): \mathcal{C} \rightarrow \mathbf{Set}$.

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 $\Delta_F: \mathcal{D}\text{-Inst} \rightarrow \mathcal{C}\text{-Inst}$.
 - Applying Δ_F to I is called “pulling I back along F ”.

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- Functor composition becomes data transformation:
 $\Delta_F: \mathcal{D}\text{-Inst} \rightarrow \mathcal{C}\text{-Inst}$.
 - Applying Δ_F to I is called “pulling I back along F ”.
 - Δ_F has two forward-directional *adjoints*, Σ_F and Π_F .
 - Let's back up a little.

The category of instances

- Given a schema $\mathcal{C}$, the *category of instances* on $\mathcal{C}$ is denoted $\mathcal{C}\text{-Inst}$.
 - The objects of $\mathcal{C}\text{-Inst}$ are functors (instances) $I: \mathcal{C} \rightarrow \mathbf{Set}$.
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- Functors between schemas allow us to move data between them.
 - Given a functor $F: \mathcal{C} \rightarrow \mathcal{D}$,
 - There are automatically three data transformation functors

$$\mathcal{C}\text{-Inst} \begin{array}{c} \xrightarrow{\Sigma_F} \\ \xleftarrow{\Delta_F} \\ \xrightarrow{\Pi_F} \end{array} \mathcal{D}\text{-Inst}$$

- Roughly: Δ =project, delete; Σ =sum, union; Π =product, join.
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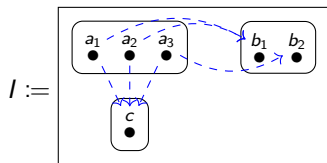
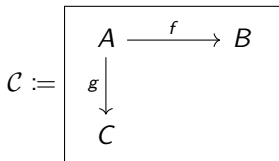
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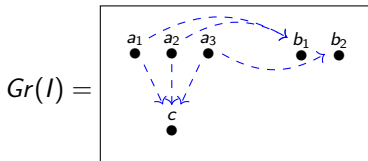
There's a lot of mathematics ready made for **hygienically** moving data.

Example: The Grothendieck construction

- Let $\mathcal{C}$ be a category and let $I: \mathcal{C} \rightarrow \mathbf{Set}$ be a functor.
- We can convert I into a category $Gr(I)$ in a canonical way:
 - Example:



- $Gr(I)$ is also known as *the category of elements of I*:

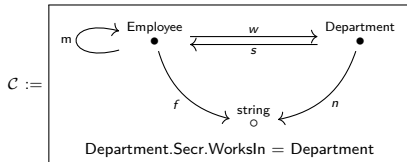


This applies to database instances

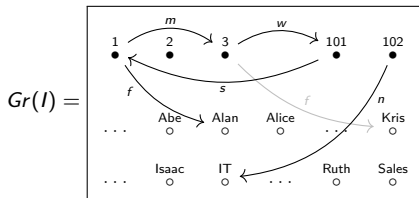
Suppose given the following instance, considered as $I: \mathcal{C} \rightarrow \mathbf{Set}$

Employee			
Id	FName	Mgr	WorksIn
1	Alan	3	101
2	Ruth	2	102
3	Kris	3	101

Department		
Id	Name	Secr
101	Sales	1
102	IT	2

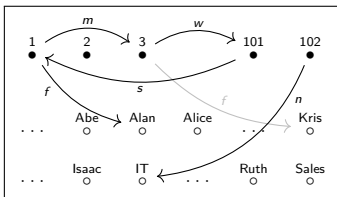


Here is $Gr(I)$, the category of elements of I :

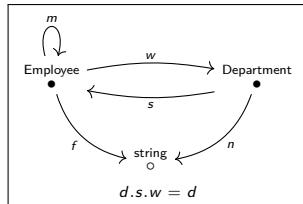


Relations to RDF

The category of elements comes with a functor $\pi: Gr(I) \rightarrow \mathcal{C}$.

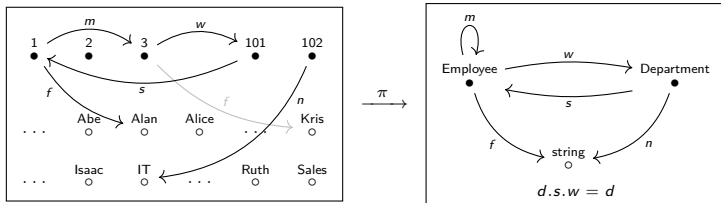


π



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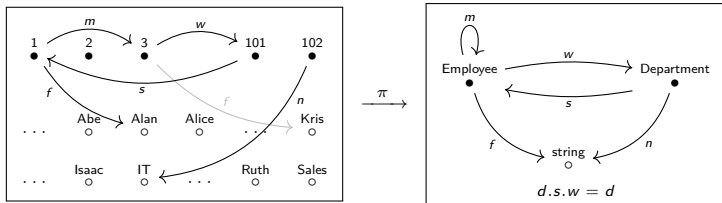


Relation to RDF triple stores and schemas:

- Each arrow $x \xrightarrow{f} y$ in $Gr(I)$ is an *RDF triple* (x, f, y) .
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 - Example: $(1, FName, Alan)$ or $(101, Secr, 1)$

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 - Example: $(1, \text{FName}, \text{Alan})$ or $(101, \text{Secr}, 1)$
- Category theoretic model of RDF
 - Think of $Gr(I)$ as RDF triple store, $\mathcal{C}$ as RDF schema.
 - SPARQL graph pattern queries fit easily into the model.
 - Models embedded dependencies (analogous to OWL schemas).

Outline

An invitation to engage with us, and solve real-world problems.

- 1 Introduction
- 2 The problem
- 3 The math
- 4 **The tool**
 - The history of AQL
 - AQL Capabilities
- 5 Conclusion

The history of AQL

- The mathematical foundations of this story are old.
 - The basic idea was known to mathematicians 60 years ago.
 - More recently we've learned a lot about how to calculate them fast.
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- We received funding from various government agencies.
 - ONR, AFOSR, NIST, NSF.
- A company spun out of MIT in 2015.
 - Categorical Informatics Inc.
 - All MIT IP is open source, all Catinf IP is not.

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- And more (natural transformations, algebraic theories, profunctors, Grothendieck construction, (co-) monads..., simply-typed lambda calculus)

Screenshot

The screenshot shows the AQL IDE interface. The main editor displays the following AQL code:

```

1 typeside Ty = empty
2
3 schema S = literal : Ty {
4   entities
5   E
6   foreign_keys
7   f : E -> E
8   path_equations
9   f.f = f
10 }
11 schema T = literal : Ty {
12   entities
13   E2
14   foreign_keys
15   f2 : E2 -> E2
16   path_equations
17   f2.f2.f2 = f2
18 }
19
20 mapping M1 = literal : S -> T {
21   entity E -> E2
22   foreign_keys f -> f2.f2
23 }
24
25 mapping M2 = literal : S -> T {
26   entity E -> E2
27   foreign_keys f -> f2
28 }
29

```

The Outline panel on the right shows the structure of the code:

- typeside Ty
 - schema S
 - schema T
 - mapping M1 : S -> T
 - mapping M2 : S -> T

A tooltip is displayed over the code at line 27, stating: "Equation $v.f.f = v.f$ translates to $v.f2.f2 = v.f2$, which is not provable". Below the tooltip, it says "Press 'F2' for focus".

The Response panel at the bottom is empty.

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- Summary of the talk

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- A simple principle—data transformations—formalized mathematically.
- And this database stuff is just one part of category theory.
 - CT has formalized the principles of mathematics, in mathematics.
 - Space, measure, operation, data, symmetry, equivalence, syntax.
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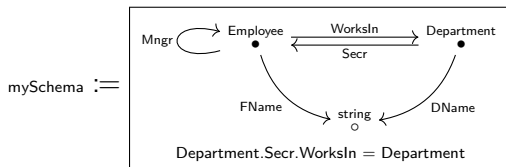
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 - There is a web of interconnection between all these principles.
- CT been recently highlighted by agencies such as NIST and DARPA.
 - CT stem cell leads to interoperability and compositionality.
 - It compresses and connects big ideas.
 - It helps you take care of all the corner cases.
 - Through strong abstraction principles, it exposes conceptual neighbors.

Summary of the talk

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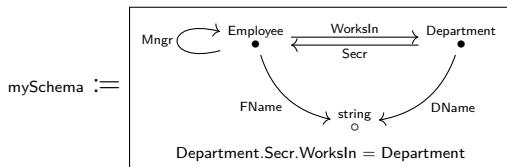
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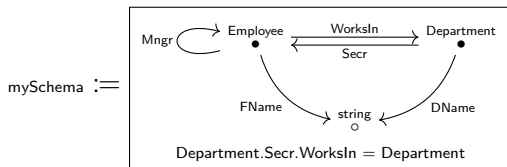
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- Information kinematics—how data moves—is well-modeled by CT.
- With a good understanding, we save a lot of time and effort.

For more...

Book: *An Invitation to Applied Category Theory: Seven Sketches in Compositionality*. Cambridge University Press, July 2019.

<https://arxiv.org/abs/1803.05316>

Company: Conexus. Website: <http://conexus.com>

Community: Category Theory Seminar, Thursdays 5:00 – 6:00, MIT Building 2, room 255. <http://brendanfong.com/seminar.html>

Conference: International Applied Category Theory Conference, July 6 – 10 at MIT. <https://www.appliedcategorytheory.org/>

Thanks for the invitation to speak!