Reglog – the game

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2019/04/11

MIT ACT seminar

Outline

Minority Report

Minority Report

The following has very little direct relation to the movie Minority Report; it's just an analogy for color.

Minority Report... regular logic style

The 2002 movie *Minority report* showed detective Tom Cruise playing seamlessly with logic.

- A computer database held relevant information.
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Let's imagine our own version of a detective scenario.

Brought to you by... regular logic – the game!

Working with logic

- Imagine you are an investigator on a case.
- You're adding to and narrowing down your set of suspects.
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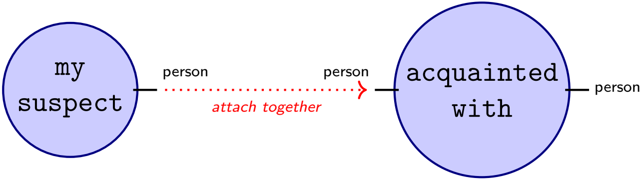
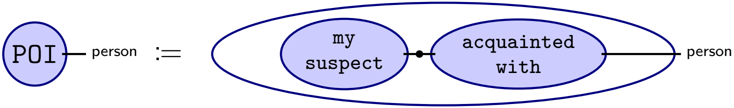
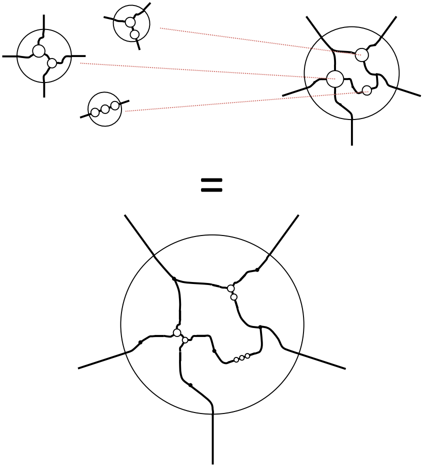


- and define POI (person of interest) as the result:



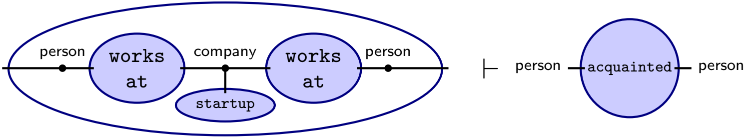
Compositionality

Of course, these concepts can be nested.



Adding beliefs

You can add beliefs about the world.



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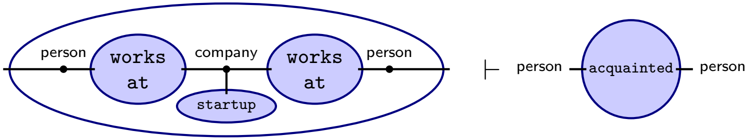

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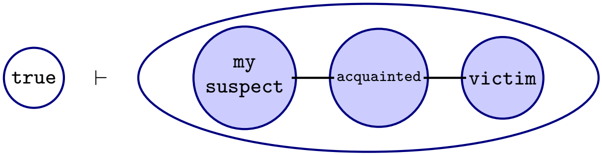
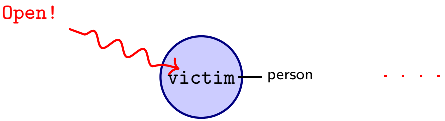


Belief: "In any case, my suspect is acquainted with the victim."

Accessing data

You can click a cell to see what's inside:



victim
person
P105
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Persons are internal identifiers; we want to see facts about the victims.

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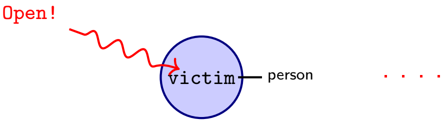

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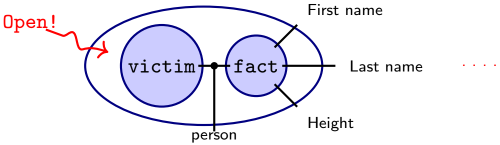

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person	First	Last	Height
P105	Alice	Hu	5'10" – 6'00"
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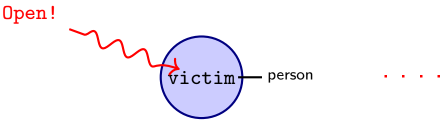

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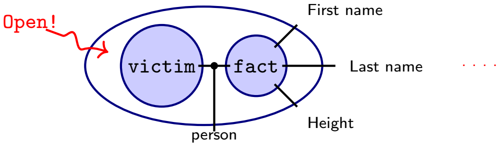

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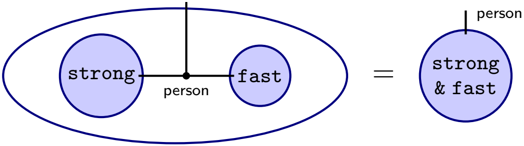
Some knowledge is missing or otherwise imperfect.

Reasoning

The machine knows basic logical reasoning.



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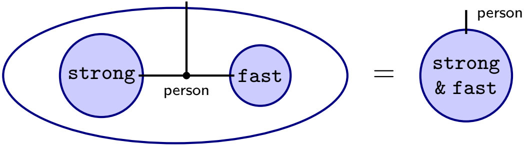

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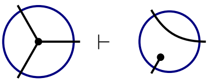
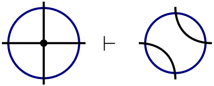
- ... breaking dots:



$\{(x, y, z) \mid x = y = z\} \subseteq \{(x, y, z) \mid x = y\}$



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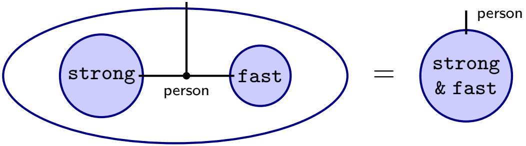

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Reasoning: regular "old" logic, with a shiny new math-specified GUI.

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You've identified certain sources of and constraints on your suspect

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The chase minimally "repairs" $I \to I'$, with I' conforming to axioms.

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Can we make this real?

Plan for rest of talk

"Detective" is not the only game in reglog – the game.

- I'll briefly discuss the mathematics involved.
- I'll talk about how it connects to the GUI described above.
- I'll end by giving several other games, besides "detective".

Outline

Regular logic and regular categories

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- Regular categories are categories $\mathcal{R}$ with
  - finite limits (terminal object 1 and pullbacks), and
  - pullback-stable image factorizations.
- Say $\mathcal{R}$ is *fully-inhabited* if $\mathcal{R}(1, r) \neq \varnothing$ for each $r \in \mathcal{R}$.
  - Set is a regular category, but not fully inhabited.
  - The category of pointed sets is fully inhabited.
  - Have categories $\mathsf{RegCat}_* \subseteq \mathsf{RegCat}$ of (fully-inhabited) regular cats.

Examples of regular categories:

- Set, and more generally any topos;
- $\mathsf{Set}^{\mathsf{op}}$, opposite of any topos, also $\mathsf{TopSp}^{\mathsf{op}}$;
- The category of models of any Lawvere theory (Groups, Rings, ...);
- The slice (also the coslice) of any regular category over any object;
- Exponential ideal: if $\mathcal{R}$ regular and $\mathcal{C}$ a category, then $\mathcal{R}^{\mathcal{C}}$ is regular.

How to think of regular categories

Regular categories $\mathcal{R}$ are those with a good *bicategory of relations*.

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Regular categories have enough structure to do regular logic.

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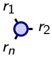
  - Have a shell 

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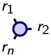
  - Have a shell 

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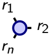
  - Have a shell 

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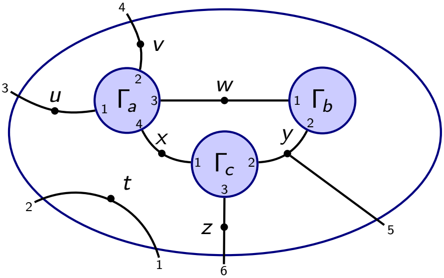
  - Wiring diagrams denote combinations of finite limits and images.
- Let's discuss wiring diagrams.

Mathematical specification of wiring diagrams

Mathematically, what is a wiring diagram $(\Gamma_a, \Gamma_b, \Gamma_c) \to \Gamma$?.



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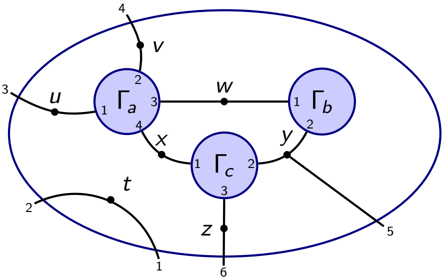
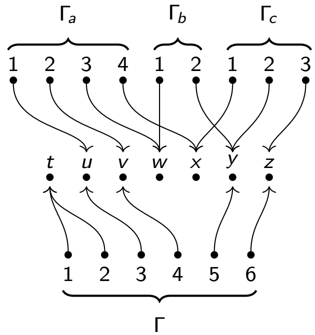


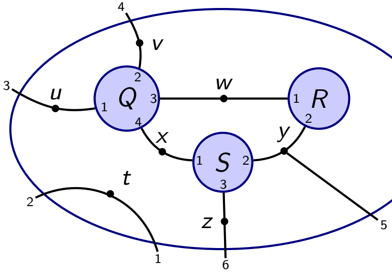


It is a morphism $(\Gamma_a, \Gamma_b, \Gamma_c) \to \Gamma$ in the operad of corelations.

- Another viewpoint: it is an equivalence relation on $\Gamma_a \sqcup \Gamma_b \sqcup \Gamma_c \sqcup \Gamma$.

Wiring diagrams as logical expressions

We can convert a wiring diagram like this into a logical expression:



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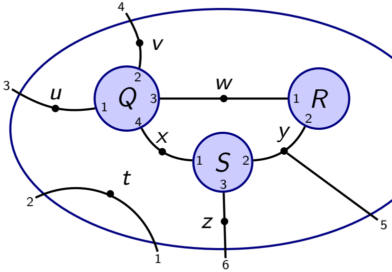

- Write type of exterior shell, naming each port by a distinct variable.
- Write quantifier $\exists(x : X)$ for each unexposed wire of type X.
- AND together internal cells, with established var. names from above.
- Equate variables for exposed ports that are connected.

$$O(t_1, t_2, u_3, v_4, y_5, z_6) := \exists(w{:}W, x{:}X).\, Q(u_3, v_4, w, x) \wedge R(w, y_5) \wedge S(x, y_5, z_6) \wedge (t_1 = t_2)$$

Formal specification of graphical calculi I

Our “minority report” detective GUI can be understood as follows.

- Fix a set T (each $t \in \mathsf{T}$ is a string label: person, height, etc.).
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  - Monoidal structure: $(1, \times)$.
- We will consider certain functors $\Phi\colon \mathbb{C}\mathsf{orel}_{\mathsf{T}} \to \mathbb{P}\mathsf{oset}$.
  - To each shell $\Gamma \in \mathbb{C}\mathsf{orel}_{\mathsf{T}}$, a poset $\Phi(\Gamma)$.
  - We denote the order in $\Phi(\Gamma)$ using the logical *entailment* symbol $\vdash$.

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We have monoidal 2-categories $\mathbb{C}\mathrm{orel}$ and $\mathbb{P}\mathrm{oset}$.

Definition

An *(inhabited) regular calculus* is a lax monoidal 2-functor

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Our terminology: *ajax* monoidal functors: the laxators

$$1 \xrightarrow{\rho_1} \Phi(0) \qquad \text{and} \qquad \Phi(v) \times \Phi(v') \xrightarrow{\rho_{v,v'}} \Phi(v + v').$$

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$$1 \underset{\lambda_1}{\overset{\rho_1}{\rightleftarrows}} \Phi(0) \qquad \text{and} \qquad \Phi(v) \times \Phi(v') \underset{\lambda_{v,v'}}{\overset{\rho_{v,v'}}{\rightleftarrows}} \Phi(v + v').$$

Regular calculi and regular categories

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A similar theorem holds when RegCat_* is replaced by RegCat: `arxiv.org/abs/1812.05765`.

Aside: operadic vs. monoidal

There are two versions of the whole story.

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- Monoidal version: more familiar, better notation.

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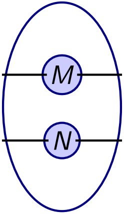
The translation uses "shells in a trench coat".



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  - Supply definition of $\varphi_1 \leq \varphi_2$, drawn perhaps as $\varphi_1 \vdash \varphi_2$.
  - To each wiring diagram w, supply monotonic function $\Phi(w)\colon \Phi(\Gamma_1) \times \cdots \times \Phi(\Gamma_n) \to \Phi(\Gamma')$.
  - Ensure Φ preserves composition, identity, and dot-breaking.

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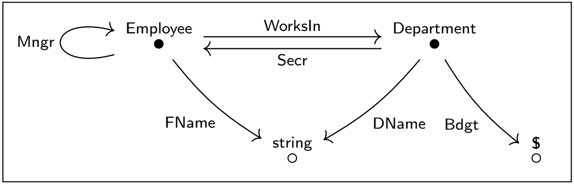

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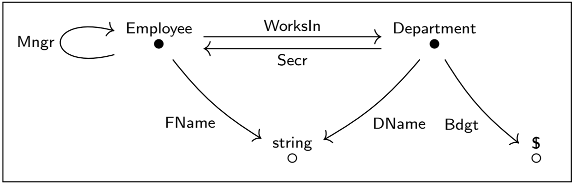

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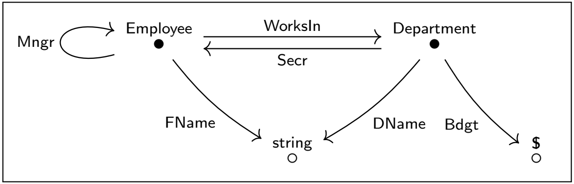

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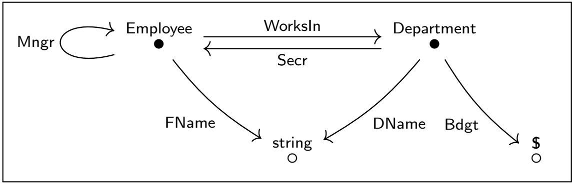

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- Each regular sequent $\phi(\vec{x}) \vdash \psi(\vec{x})$ is called an embedded dependency.
- Chase these EDs to "repair data", forcing the axioms to hold.

Outline

Common interface

"Reglog – the game" is actually a bunch of games.

- In common: same GUI and common forms of interaction
  - Create new shells, attach shells, compose wiring diagrams
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- Some examples:
    - Where are my keys?
    - Smart witter
    - Partitions game
    - Boolean circuits
    - Solve equations
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Game: Where are my keys?

A simpler version of detective game: no reasoning $\vdash$, just query.

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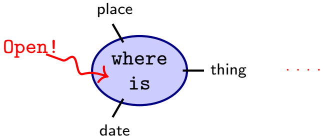

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thing	place	date
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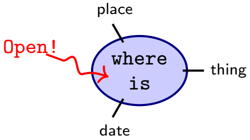

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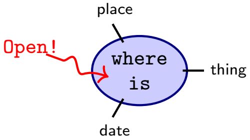

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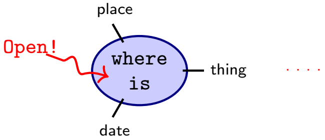

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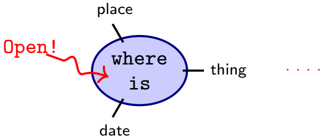

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  - There is an internet-published "doctor" cell  .
  - Connect specialty –cardiologist to doctor's speciality port.
  - Connect its availability and insurance ports to your own.
  - Output doctor's name and phone number.

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A smarter social network.

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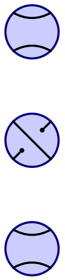
  - Same idea for human concepts; find reusable ideas (memes).

Game: Partitions puzzle

Put pieces:



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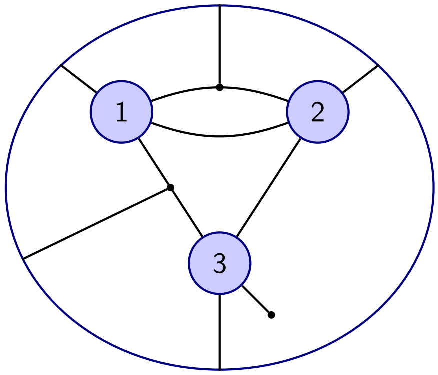

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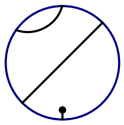

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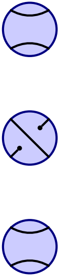
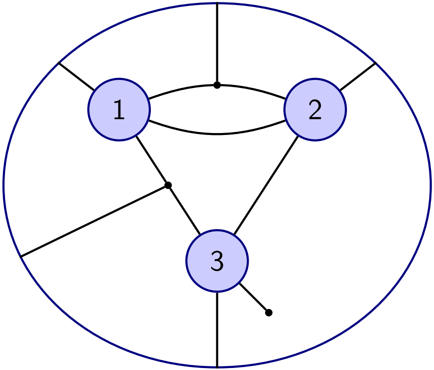
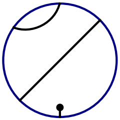
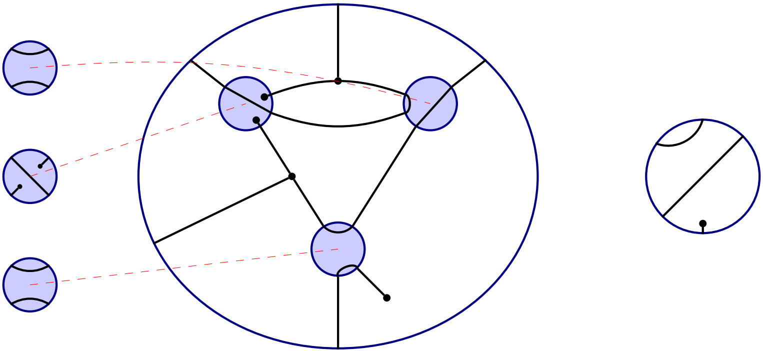






I'll give you a minute to solve it.

Game: partitions puzzle (solution)



Game: Boolean circuits

Boolean circuits are special cases of boolean relations.

- What are boolean relations?
  - A boolean relation is a subset of $\mathbb{B}^n = \{\texttt{true}, \texttt{false}\}^n$ for some n.
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Game: Solving equations

Consider an arbitrary system of equations having the following form:

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\begin{aligned}
f_1(\mathbf{t}, u, \mathbf{v}) &= 0 \\
f_2(\mathbf{v}, w, x) &= 0 \\
f_3(u, w, x, y) &= 0 \\
f_4(x, \mathbf{z}) &= 0
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Bold variables are those we want to *expose*; others are latent or *unexposed*.

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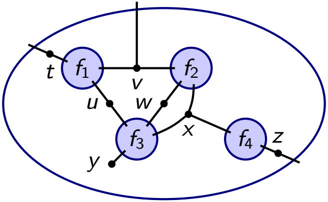

Said another way, we want $\{(t, v, z) \mid \exists u, w, x, y : f_1 = f_2 = f_3 = f_4 = 0\}$.

Systems of equations via pixel arrays

Consider just two equations $f(x, w) = 0$ and $g(x, y) = 0$.

- Plot each in its own bounding box, say in the range $[-1.5, 1.5]$.
- Consider the plots as matrices M, N whose entries are on/off pixels.
- That is, M and N are *boolean matrices* corresponding to f and g.

Multiplying these two matrices MN yields the simultaneous solution.

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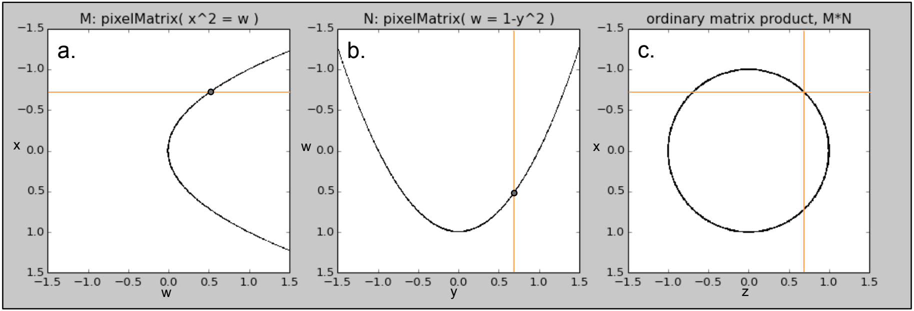
- For example, plot equations $x^2=w$ and $w=1-y^2$, and multiply.



A more complex example

The following eq's are not differentiable, nor even defined everywhere.

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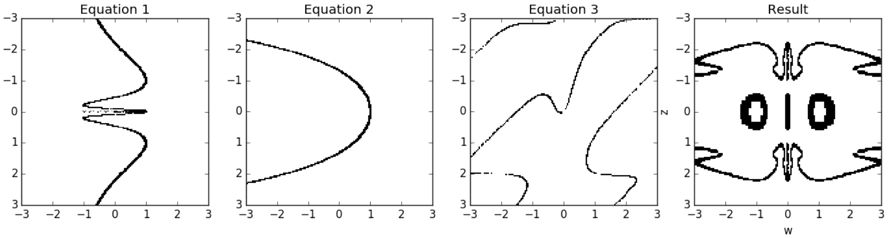
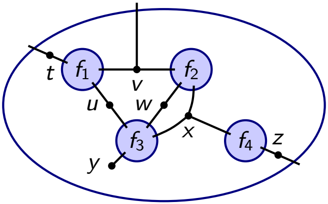


Game: systems of linear equations

PA is the backend for a game that plots sol'ns to arbitrary systems.



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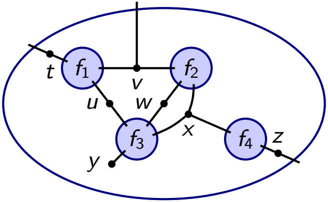

- Similar game: each cell f_i is a linear eq'n, e.g. $x_1 + 3x_2 - 2x_4 = 0$.
- Then the outer cell is a linear equation too.
- "Exponentially" smaller matrices to multiply.

Game: theory of a group, ring, etc.

Monoids, groups, rings, modules: each has an associated algebraic theory.

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  - Let's say the theory of monoids.
    - Have a multiplication cell 

 and a unit cell 

    - These satisfy various equations (more generally, regular axioms)

Game: theory of a group, ring, etc.

Monoids, groups, rings, modules: each has an associated algebraic theory.

- Choose any algebraic theory, or more generally regular theory.
  - Let's say the theory of monoids.
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  - Like in Smart witter, machine could learn common moves.
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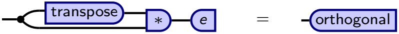

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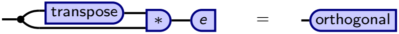
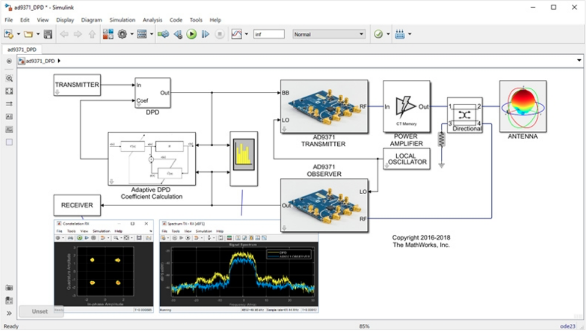
- Within the game, create new cells and manipulate them.
  - E.g. add orthogonal and transpose and axiomatize them.



Like "fold-it" for protein folding, players can help w/o understanding.

Game: Simulink

- Simulink (makers of Matlab): model and simulate dynamic systems.



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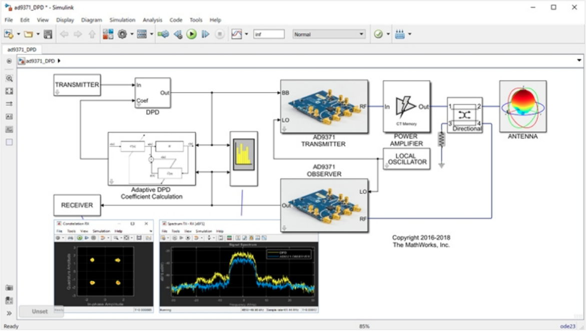

- Connect up smaller dynamic systems.
- Each can be understood as a relation in the temporal topos (TTT).
- Use reglog – the game as an interface for Simulink.

Outline

Let's make this real!

We're ready to make this happen.

- The background math is complete
  - We understand the data structures involved.
  - Experience shows that coding it will uncover hidden assumptions.
  - Example: maybe player can "transform" typeset T and Φ, mid-game.
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