Applied Category Theory:
Towards a science of interdisciplinarity

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Outline

A road to true interdisciplinarity

- Scientific disciplines are conceptual analogies of the world.
  - Science: a schematic, conceptual account of phenomena.
  - Engineering is using these accounts to channel world events.
  - But how do different disciplines and accounts cohere?
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  - Interdisciplinarity consists of effective analogy-making.
  - To go further, we need to formalize the analogies themselves.
- Better yet: we need a conceptual stem-cell.
  - Something that can differentiate into huge variety of forms.
  - Find the analogies between forms as aspects within the stem cell.

Hard science requires math

Consider as you watch:

- Is a hard science of interdisciplinarity & interoperability possible?

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- All forms of pure math... (we'll briefly discuss this)
- Databases and knowledge representation (categories and functors)
- Functional programming languages (cartesian closed categories)
- Universal algebra (finite-product categories)
- Dynamical systems and fractals (operad-algebras, co-algebras)
- Shannon Entropy (operad of simplices)
- Partially-ordered sets and metric spaces (enriched categories)
- Higher order logic (toposes = categories of sheaves)
- Measurements of diversity in populations (magnitude of categories)
- Collaborative design (enriched categories and profunctors)
- Petri nets and chemical reaction networks (monoidal categories)
- Quantum processes and NLP (compact closed categories)

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  - Conclude that math/CT explains everything and hence nothing?

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  - Conclude that math/CT explains everything and hence nothing?
- Stem cells don't do work until they differentiate.
  - "Adult-level" work requires differentiation and optimization.
  - But the unified origins lead to impressive interoperability.

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- Designed to transport theorems from one area of math to another.
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- And it's branched out from math in a big way.
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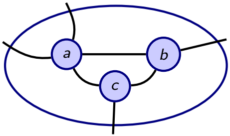
Compositionality: nestable arrangements



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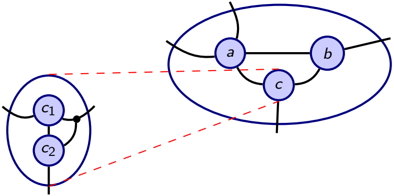

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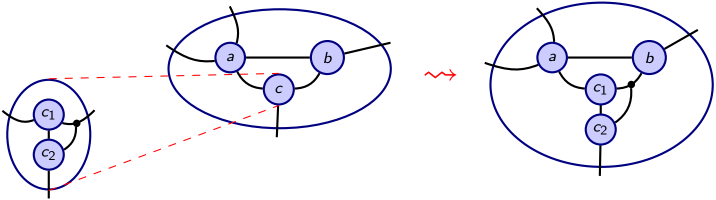

What are compositional operations?

An operad consists of:

- A collection of sorts $X, Y, \ldots$,
- And ways to arrange them, $\varphi\colon X_1, \ldots, X_k \to Y$,
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Ask me now or later, if you want a more formal definition.

Operads are everywhere

Operads are used unconsciously in many fields.

- Electrical engineering: “wiring diagrams”
- Design: “set-based design”
- Computer programming: “data flow”
- Natural language processing: “grammars”
- Materials science: “hierarchical materials”
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- Operads structure this sort of thinking.
- With mathematical structure, we can go much further.

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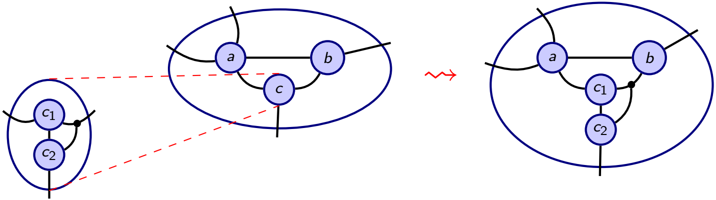
Let's look for sorts, arrangements, and nesting in some examples.

Operad 1: wiring diagrams

Sorts: circles with ports. Arrangements: wiring diagrams.



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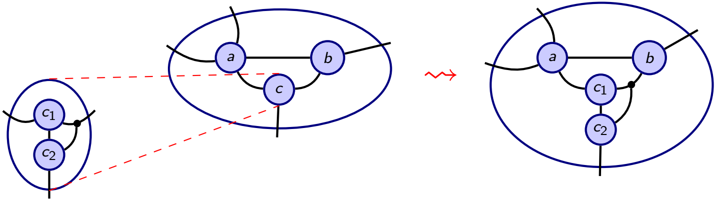

Applications: tensor networks, databases, contracts.

Operad 2: another kind of wiring diagram

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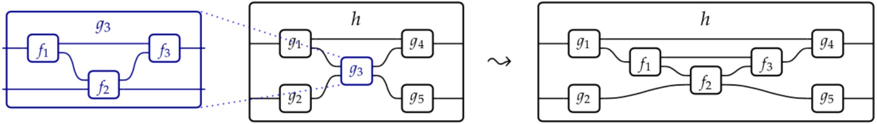

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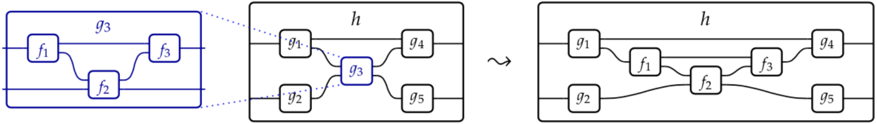

Applications: dynamical systems, control theory.

Operad 2: hierarchical protein materials

There is an operad $\mathcal{M}$ for composing hierarchical protein materials.

- A protein is an arrangement of simpler proteins.
  - There are "atomic" proteins: amino acids.
  - Protein materials include your skin: stretchable, breathable, waterproof.
  - Materials scientists would *love* to make materials like this.

[1] Giesa, T.; Jagadeesan, R.; Spivak, D.I.; Buehler, M.J. (2015) "Matriarch: a Python library for materials architecture." *ACS Biomaterials Science & Engineering*.

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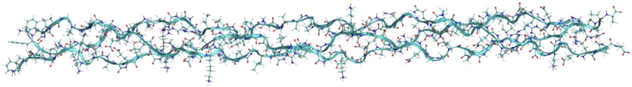
- Assemble new proteins from old:
  - arrange in series or parallel (H-bonds), or
  - arrange in helices, double helices, any conceivable curve, etc.



- Collagen has a nested structure: it is an array, each fiber of which is a triple helix, each strand of which is a helix, each unit of which is an amino acid.[1]

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Operad 3: probabilities

Even an operad with one sort can be interesting.

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- Say sorts={event}, i.e. "event" is the only sort in $\mathcal{P}$.
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Example: think arrangements = probability distributions.

- Let $\mathcal{P}_k := \left\{(x_1, \ldots, x_k) \in \mathbb{R}^k_{\geq 0} \;\middle|\; x_1 + \cdots + x_k = 1\right\}$.

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    - The event "flip coin" as arrangement of two sub-events
  - roll die: $r = (\frac{1}{6}, \ldots, \frac{1}{6}) \in \mathcal{P}_6$.
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- The nesting rule is multiplication:
  - Flip a coin: result decides whether to roll a die or pick a card.

$$f \circ (r, p) = \Bigg(\underbrace{\frac{1}{12}, \ldots, \frac{1}{12}}_{6 \text{ times}}, \underbrace{\frac{1}{104}, \ldots, \frac{1}{104}}_{52 \text{ times}}\Bigg) \in \mathcal{P}_{58}$$

A zoo of operads: Grammars

Any context-free grammar is an operad.

⟨sentence⟩	::=	⟨noun-phrase⟩ ⟨verb-phrase⟩
⟨noun-phrase⟩	::=	⟨pronoun⟩ | ⟨proper-noun⟩ | ⟨determiner⟩ ⟨nominal⟩
⟨nominal⟩	::=	⟨noun⟩ | ⟨nominal⟩ ⟨noun⟩
⟨verb-phrase⟩	::=	⟨verb⟩ | ⟨verb⟩ ⟨noun-phrase⟩ | ⟨verb⟩ ⟨prep-phrase⟩
⟨prep-phrase⟩	::=	⟨preposition⟩ ⟨noun-phrase⟩

Of course, it doesn't have to be English

- Grammars are used to design custom languages.
- Legal arguments, arithmetic expressions, programming languages.

And how is this an operad?

- The sorts are the parts of speech.
- The arrangements are the production rules.
- Nesting is given by "syntax trees".

The operad of sets

There is an operad **Set** of sets and functions.

- The sorts in **Set** are all the sets, e.g. $\{\mathtt{r}, \mathtt{g}, \mathtt{b}\}$, $\mathbb{N}$, $\mathbb{R}^2$, etc.
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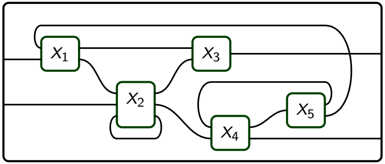
Think of sets as containing features one can sort into.

- An arrangement is a clustering of a k-dimensional feature space.

Mode-dependent networks of dynamical systems

Recall the wiring diagrams operad, used in dynamical systems.



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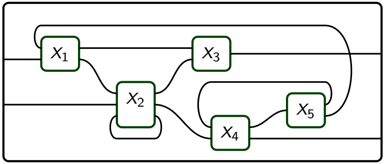

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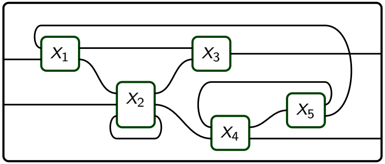

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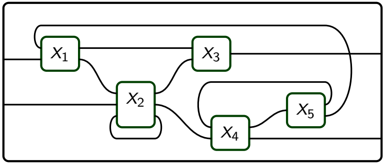

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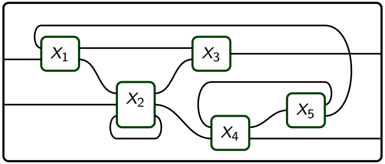

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As boxes send signals and update their states, their modes change, and hence their arrangement does too.

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- People are already using them in many different fields.
- We can describe them all in a common language.
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