Monadic Decision Processes for Hierarchical Planning

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I. Introduction

  A. Hierarchical planning — 3 mins
  B. Markov decision processes — 3 mins
    1. Generalization: $\Sigma \times S \to \text{Dist}(S \times \mathbb{R})$
    2. Reinforcement learning
    3. We won't do any optimization today, just understand framework
  C. Plan — 2 mins
    1. Background
    2. M-MDPs via monoids and monoidal monads
    3. Category of M-MDPs for hierarchical planning

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II. Background

  A. Monoids and actions — 3 mins
    1. Monoid $\mathcal{A}$ as category with one object
    2. Functor $\mathcal{A} \to \mathsf{Set}$ as action
  B. Monads on $\mathcal{C}$
    1. Definition — 2 mins
    2. Examples: — 5 mins
      a. List
      b. Dist
      c. $\mathbb{R}_{\geq 0}^{\oplus X}$ "populations"="formal $\mathbb{R}$-linear combinations of elements in X", e.g. $3x_1 + 4.7x_4$.
      d. $X \mapsto X \times R$ for any monoid R
      e. $X \mapsto X + E$ for any set E
      f. $X \mapsto X^T$ for any set T
      g. $X \mapsto \text{Dist}(X \times \mathbb{R})$
  C. Kleisli category Kls_M
    1. Definition — 3 mins
    2. MDPs as maps in $\mathsf{Kls}_{\text{Dist}(-\times\mathbb{R})}$ — 2 mins

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III. M-DPs via monoids and monoidal monads

A. MDPs as functors $T\colon \mathcal{A} \to \mathsf{Kls}_{\mathrm{Dist}(-\times\mathbb{R})}$ — 3 mins

1. Unpacking:
   a. a set S
   b. a monoid $(A, *, e)$,
   c. a function $T\colon A \to \mathsf{Set}(S, M(S))$
   d. i.e. a function $T\colon A \times S \to M(S)$
   e. satisfying

$$\begin{array}{ccc} S & \xrightarrow{\eta} & MS \\ {\scriptstyle\cong}\downarrow & & \uparrow{\scriptstyle T} \\ \underline{1} \times S & \xrightarrow[e\times S]{} & A \times S \end{array} \qquad \begin{array}{ccccc} A \times A \times S & \xrightarrow{*\times\underline{1}} & A \times S & \xrightarrow{T} & MS \\ {\scriptstyle e\times T}\downarrow & & & & \uparrow{\scriptstyle\mu} \\ A \times MS & \xrightarrow[\gamma_{A,S}]{} & M(A \times S) & \xrightarrow[MT]{} & M^2S \end{array}$$

2. Case $\mathcal{A} = \mathrm{List}(\Sigma)$ — 2 mins
   a. Then T is equivalently a set S and a function $\Sigma \times S \to M(S)$.
3. $\mathcal{A}$ is not necessarily free — 2 mins
   a. Eg. $\mathcal{A} = \mathbb{Z} \times \mathbb{Z}$ grid walk is a group.
   b. Want to ensure $T(a * a')$ is "do a then do a'.
   c. E.g. want to ensure the undo operation really does undo.

B. Other monads — 4 mins

1. Identity
2. Deterministic with reward
3. No rewards, other kinds of rewards
4. Populations
5. Possibilistic rather than probabilistic
6. Exceptions
7. Tasks $X \mapsto X^T$

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IV. Maps of M-DPs for hierarchical planning

A. Morphisms

1. Given $T'\colon A' \times S' \to M(S)$ and $T\colon A \times S \to M(S)$, what's a map? — 1 min
2. One answer in two pieces — 2 mins
   a. "a coarse-graining" $g\colon S' \to S$ and
   b. "for every high-level action and low-level state, a low-level action", a planner: $p\colon A \times S' \to A'$.
   c. This is called a lens (or bilens) $\binom{S'}{A'} \to \binom{S}{A}$.
3. How should A' relate to A?

a. The obvious looks **too strong / unrealistic**. 3 mins

$$\begin{array}{ccccccc} A \times S' & \xrightarrow{A \times \Delta_{S'}} & A \times S' \times S' & \xrightarrow{p \times S'} & A' \times S' & \xrightarrow{T'} & M(S') \\ {\scriptstyle A \times g}\downarrow & & & & & & \downarrow{\scriptstyle M(g)} \\ A \times S & & & \xrightarrow[T]{} & & & M(S) \end{array} \tag{1}$$

b. Alternative: 5 mins

(1) Assume M monoidal

(2) Keep $g \colon S' \to S$ and $p \colon A \times S' \to A'$.

(3) Add "sampling" map $\sigma \colon S \to M(S')$ with $\sigma \mathbin{⨾} M(g) = \eta$, and

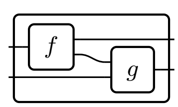
(4) ask the following to commute:

$$\begin{array}{ccccccc} A \times S & & & \xrightarrow{T} & & & M(S) \\ {\scriptstyle A \times \sigma}\downarrow & & & & & & \uparrow{\scriptstyle M(g)} \\ A \times M(S') & \to & M(A \times S' \times S') & \xrightarrow[M(p \times S')]{} & M(A' \times S') & \xrightarrow[\mu(M(T'))]{} & M(S') \end{array}$$

(5) It is tedious but "straightforward" to check that this is a cat.

B. Symmetric monoidal structure

1. M-MDP is a symmetric monoidal category. 3 mins
   a. Given $\mathcal{A} \to \mathsf{Kls}_M$ and $\mathcal{B} \to \mathsf{Kls}_M$
   b. Get $\mathcal{A} \times \mathcal{B} \to \mathsf{Kls}_M \times \mathsf{Kls}_M \xrightarrow{\otimes} \mathsf{Kls}_M$.
   c. Idea: two MDPs in parallel
2. Wiring diagram 4 mins



   a. Each wire is an M-MDP
   b. Box: how to perform any tuple of high-level actions as tuple of low-level actions
   c. WD: imagine low-level states at left, high-level actions right

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