Graphical abelian logic

David I. Spivak* and Brendan Fong

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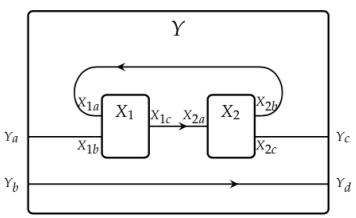

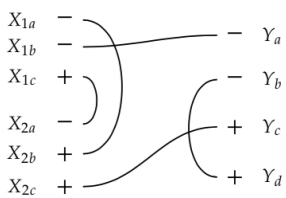

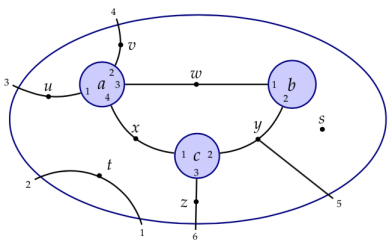

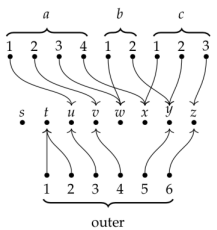

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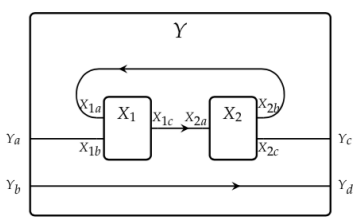

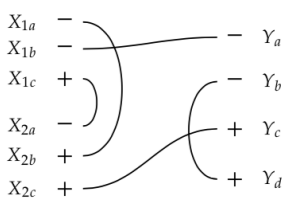

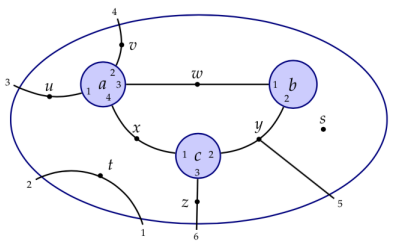

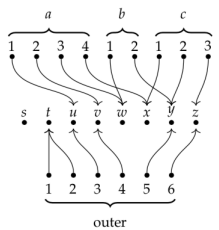

- Brendan talked about how to get regular categories this way.
- Today: abelian categories this way.

Abelian categories

Definition

A category $\mathcal{A}$ is *abelian* if

- it has a zero object 0;
- every pair of objects has a product and a coproduct;
- every morphism has a kernel and a cokernel; and
- every monomorphism is a kernel and every epimorphism is a cokernel.

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Just think fgAb or $\mathrm{Vect}_{\mathbb{R}}$.

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- The elements of A/Q are those of A, except with Q-noise added.
- Anonymize / wash out the Q dimension.

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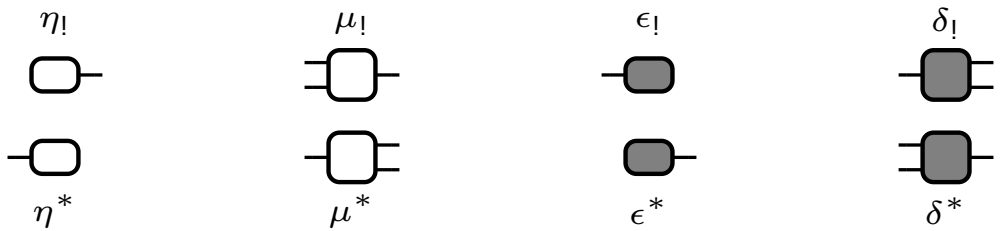

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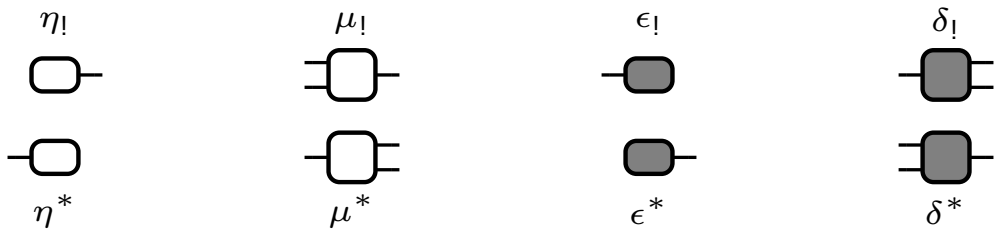

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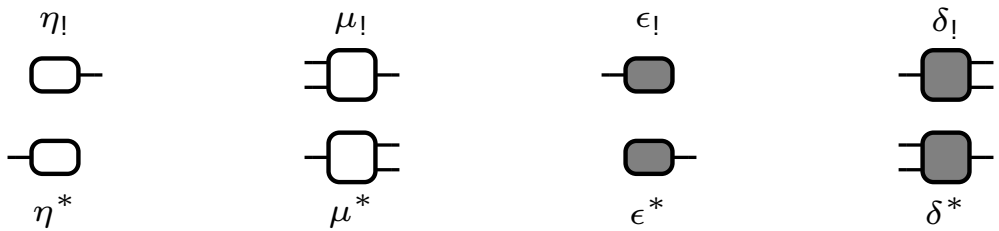

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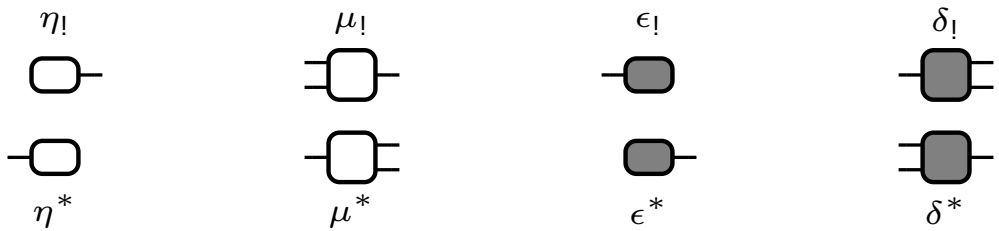

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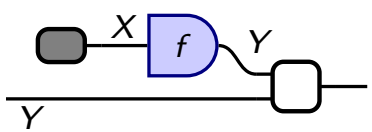
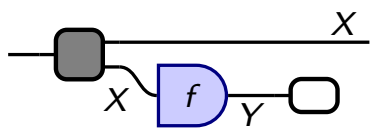
$Y \to \operatorname{coker}(f)$



$\ker(f) \to X$



The po-prop of abelian relations

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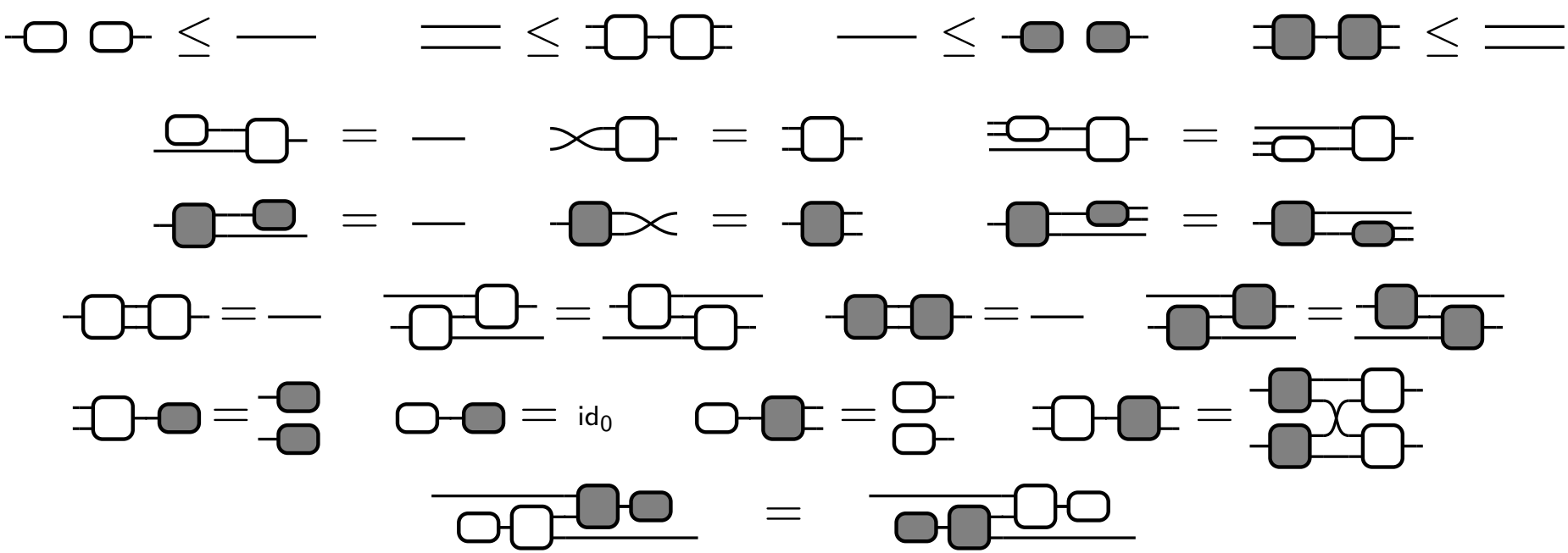
such that: $(\eta_!, \mu_!, \eta^*, \mu^*)$ is an adjoint Frobenius monoid; $(\epsilon_!, \delta_!, \epsilon^*, \mu^*)$ is an adjoint Frobenius comonoid,

the left (iff right) adjoints form a bimonoid, mediating isomorphism is an involution:



Drawing functors $P\colon \mathbb{A} \to \mathbb{P}\mathrm{oset}$

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Main interest: lax monoidal po-functors $P\colon \mathbb{A} \to \mathbb{P}\mathrm{oset}$.

- String diagrams from $\mathbb{A}$ govern abelian categories.
- I.e. functors $P\colon \mathbb{A} \to \mathbb{P}\mathrm{oset}$ correspond to abelian cats.

Example: Finitely-generated abelian groups

Let's use fgAb to build an example lax monoidal po-functor $P\colon \mathbb{A} \to \mathbb{P}\mathrm{oset}$.

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- Define $P(\delta_!)\colon \mathrm{Sub}(\mathbb{Z}) \to \mathrm{Sub}(\mathbb{Z}^2)$ by $R \mapsto \{(x, x) \mid x \in R\}$.
- Define $P(\epsilon^*)\colon 1 \to \mathrm{Sub}(\mathbb{Z})$ by $1 \mapsto \top$.
- Define $P(\delta^*)\colon \mathrm{Sub}(\mathbb{Z}^2) \to \mathrm{Sub}(\mathbb{Z})$ by $R \mapsto \{x \mid (x, x) \in R\}$.

$\mathrm{Sub}(\mathbb{Z}^-)\colon \mathbb{A} \to \mathbb{P}\mathrm{oset}$ is a lax monoidal po-functor

The assignment $P(n) \coloneqq \mathrm{Sub}(\mathbb{Z}^n)$ as above is a lax monoidal po-functor.

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  - Product gives a map $\times\colon \mathrm{Sub}(\mathbb{Z}^m) \times \mathrm{Sub}(\mathbb{Z}^n) \to \mathrm{Sub}(\mathbb{Z}^{m+n})$.
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- $\mathrm{Sub}(\mathbb{Z}^-)$ is 2-functorial (preserves equations and ineq's).

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$\mathrm{Sub}(\mathbb{Z}^-)\colon \mathbb{A} \to \mathbb{P}\mathrm{oset}$ is in fact *bi-ajax* (bi-adjoint lax monoidal).

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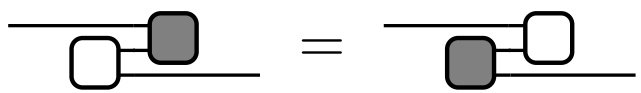
- $\mathbb{A}$ has a "negation involution"  .

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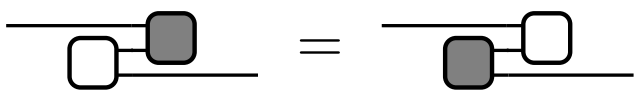
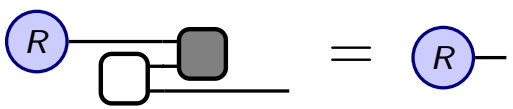
- $\mathbb{A}$ has a "negation involution" 
- $\mathrm{Sub}(\mathbb{Z}^-)$ applied to the negation involution sends $R \mapsto \{x \mid -x \in R\}$.
- Subspaces of $\mathbb{Z}^n$ are closed under negation, so this is identity.
- $\mathrm{Sub}(\mathbb{Z}^-)$ applied to negation involution in $\mathbb{A}$ is identity in $\mathbb{P}\mathrm{oset}$.



Main theorem

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An *abelian calculus* is a pair $(\mathcal{C}, P)$, where $\mathcal{C}$ supplies abelian relations and $P\colon \mathcal{C} \to \mathbb{P}\mathrm{oset}$ is bi-ajax and preserves involutions.

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Abelian categories are reflective in the 2-category of abelian calculi.

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Thanks! Questions and comments welcome!