Addressing hypercomplexity: The ACT pipeline from abstraction to application

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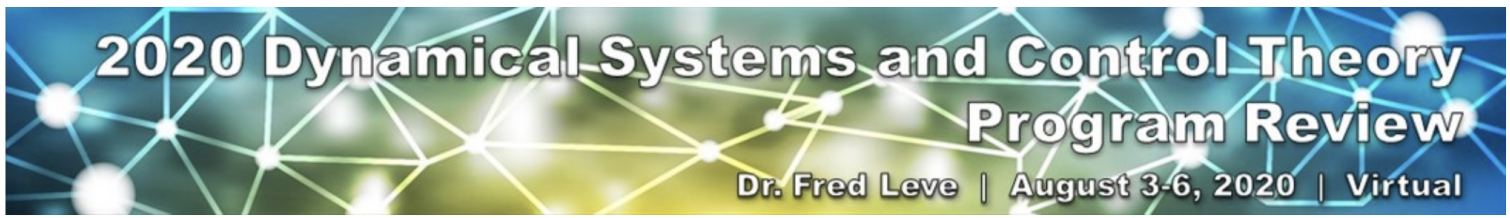
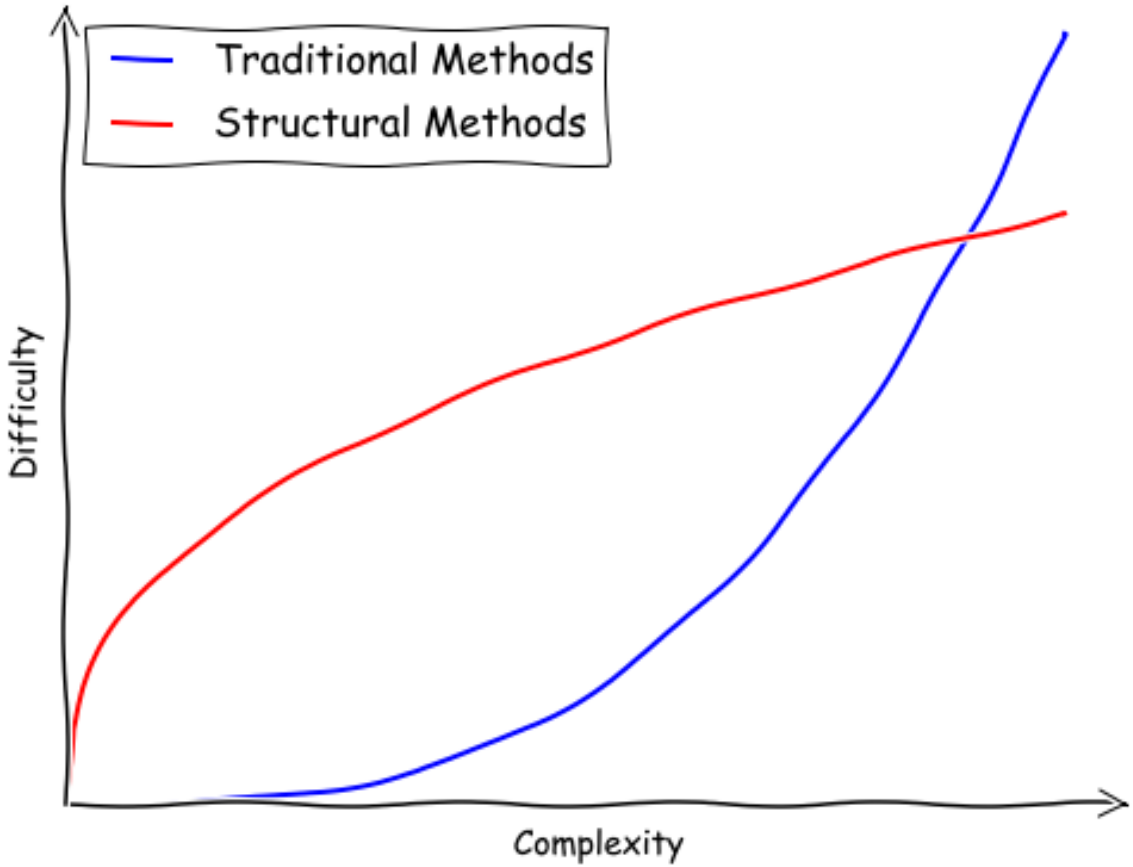


Outline

What's wrong?

As problem complexity increases, good organization becomes crucial.



(Courtesy of Spencer Breiner, NIST)

The problem is hypercomplexity

The world is becoming more complex.

- We need to deal with more and more diverse sorts of interaction.
- Rather than closed dynamical systems, everything is open.
- The state of one system affects those of systems in its environment.

Examples are everywhere.

- Designing anything requires input from more diverse stakeholders.
- Software is developed by larger teams, touching a common codebase.
- Scientists need to use experimental data obtained by their peers.
  - (But the data structure and assumptions don't simply line up.)

We need to handle this new complexity; traditional methods are faltering.

How might we deal with hypercomplexity?

Organize the problem spaces effectively, as objects in their own right.

- Find commonalities in the various problems we work on.
- Consider the ways we build complex problems out of simple ones.
- Think of *problem spaces and solutions* as mathematical entities.

With that in hand, how do you solve a hypercomplex problem?

- Migrating subproblems to groups who can solve them.
- Take the returned solutions and fit them together.
- Ensure in advance that these solutions will compose.
  - Do this according the mathematical problem-space structure.

Applied category theory

Applied category theory is a growing field.

- Current well-known applications
  - Pure math, where it's been revolutionary.
  - Quantum compilers (Cambridge Quantum Computing)
  - Database integration (Conexus AI)
  - High assurance programming (Haskell)
- Less-known applications
  - Robotics design (Andrea Censi's co-design language)
  - Model-aware scientific computing (Semanticmodels.jl)

We can use ACT to structure problem / solution spaces. Example:

- "Indexed monoidal double categories" for open dynamical systems.
- Organizes trajectories, steady states, periodic orbits, ...
- ... and what happens to them as we wire open systems together.

The problem holding ACT back

Applied category theory is very abstract.

- This is both a blessing and a curse.
  - Blessing: it applies to a very large swath of science and tech.
  - Curse: it requires restarting from the ground up.
- Coming out of pure math, there hasn't been sufficient drive for tools.
- We need more "applied applied category theory".

We're still at the very beginning of what's possible.

ACT Pipeline: from concrete to theoretical

To bring category theory to the real world, we must bridge the gap.

- We need powerful abstractions—pure theory—to guide applications.
- We need real-world applications to guide theory.

My group aims to span this spectrum.

- Find common core abstractions, aiming for elegance and simplicity.
- Apply it to the real world and see what else is needed.

We work with mathematicians, scientists, and engineers of all sorts.

Plan of the talk

- Discuss some recent work of our group;
- Discuss some early work in building tools for ACT;
- Conclude.

Outline

Perception pipeline for robotics applications

Joint with Pasquale Antonante and Luca Carlone (LIDS @ MIT).

- Domain: robotics and autonomous vehicle perception.
- Problem: integrate information from many different subsystems.
  - LIDAR, IMU, Camera, high-definition map, GPS.
  - What happens when they disagree?
- Previous solution: diagnosability graph to detect faults.
- Our solution: use temporal type theory to bring in time.
  - Get a graph for every *interval* rather than single slice of time.
  - We proved that this improves fault detection and diagnosability.

Not a huge result, but both sides benefited from knowledge exchange.

Temporal type theory and stochastic processes

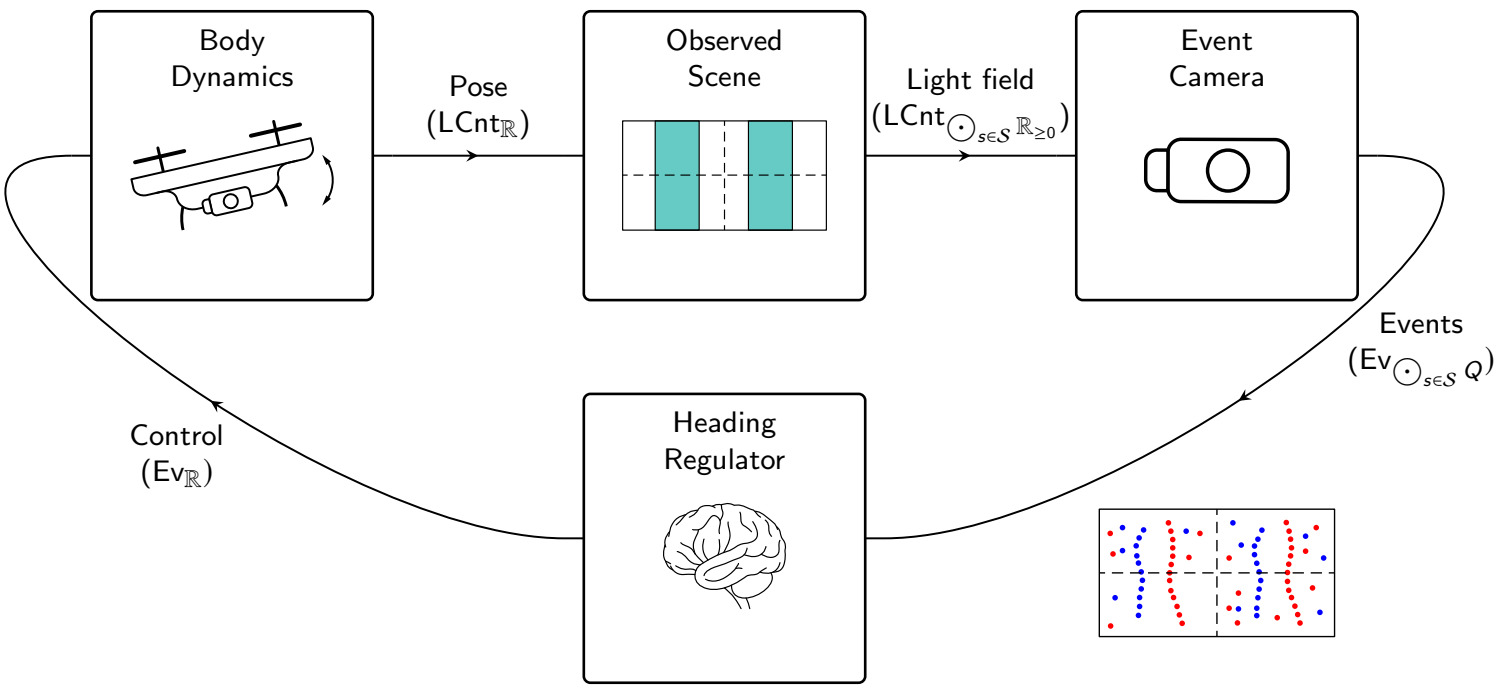
- Previously we developed a sheaf-theoretic system for general behavior.
  - Embed differential equations, inclusions, etc.
  - Discrete systems, hybrid systems, etc.
  - Delays, non-determinism, etc.
  - A model-agnostic "big tent" for anything that behaves in time.
  - A dependent type theory and logic for combining behaviors.

  Joint with Tobias Fritz (Perimeter Institute)
- Wanted to include measure theory, to consider behavioral tendencies.
  - State axioms of probability valuations using the internal language.
  - (Used the space of compacts, as in Helene Frankowska's talk.)
  - Result: a probability theory takes place in the local type theory.
  - The result generalizes stochastic processes to any spacetime.

Event-based systems

Joint with Gioele Zardini, Andrea Censi, Emilio Frazzoli (IDSC @ ETH)

- Used temporal type theory to combine the parts of an UAV.
- In particular: event-based system, e.g. event cameras.
- Any finite number of different events in a given time interval.



Outline

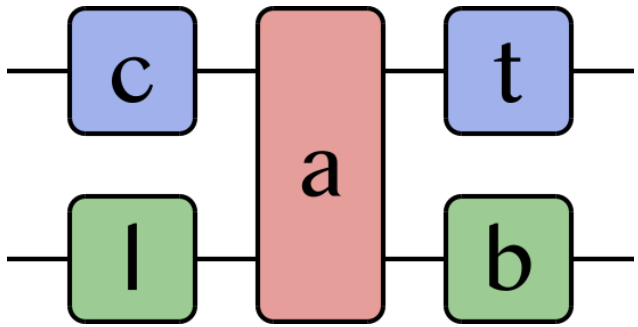
How to implement scientific and engineering models

Joint work with Evan Patterson (Stanford)

- **Numerical code**: C/C++/Fortran or indirectly via Python, R, ...
  - no high-level representations: new model $\implies$ new code
  - implementation is challenging and error prone
- **Domain-specific languages**: Simulink, Stan, TensorFlow, ...
  - easier to build new models, but only within confines of the DSL
  - abstractions and algorithms cannot be shared across domains
- **Computer algebra systems**: Mathematica, Maple, ...
  - powerful symbolic representations, but limited numerics
  - mainly confined to calculus and algebra

Catlab: a fresh approach to scientific modeling

This trichotomy is a false one.



Catlab.jl is a new framework for applied category theory:

- Written in **Julia**, which solves the "two-language problem":
  - A modern, high-level programming language
  - Just-in-time compiler generates fast code for numerical computing
- Catlab is a **hybrid system**:
  - Abstractions from category theory allow functionality to be shared between different domains and embedded DSLs
  - Symbolic and diagrammatic computing for categorical algebra
  - Numerical code generation and integration with Julia ecosystem

Case study: $\mathcal{C}$-sets as a data structure

A *$\mathcal{C}$-set* (copresheaf) is a functor $\mathcal{C} \to$ **Set**.

Things that are $\mathcal{C}$-sets: graphs, simplicial sets, Petri nets, wiring diagrams, databases with domain-specific schemas, ...

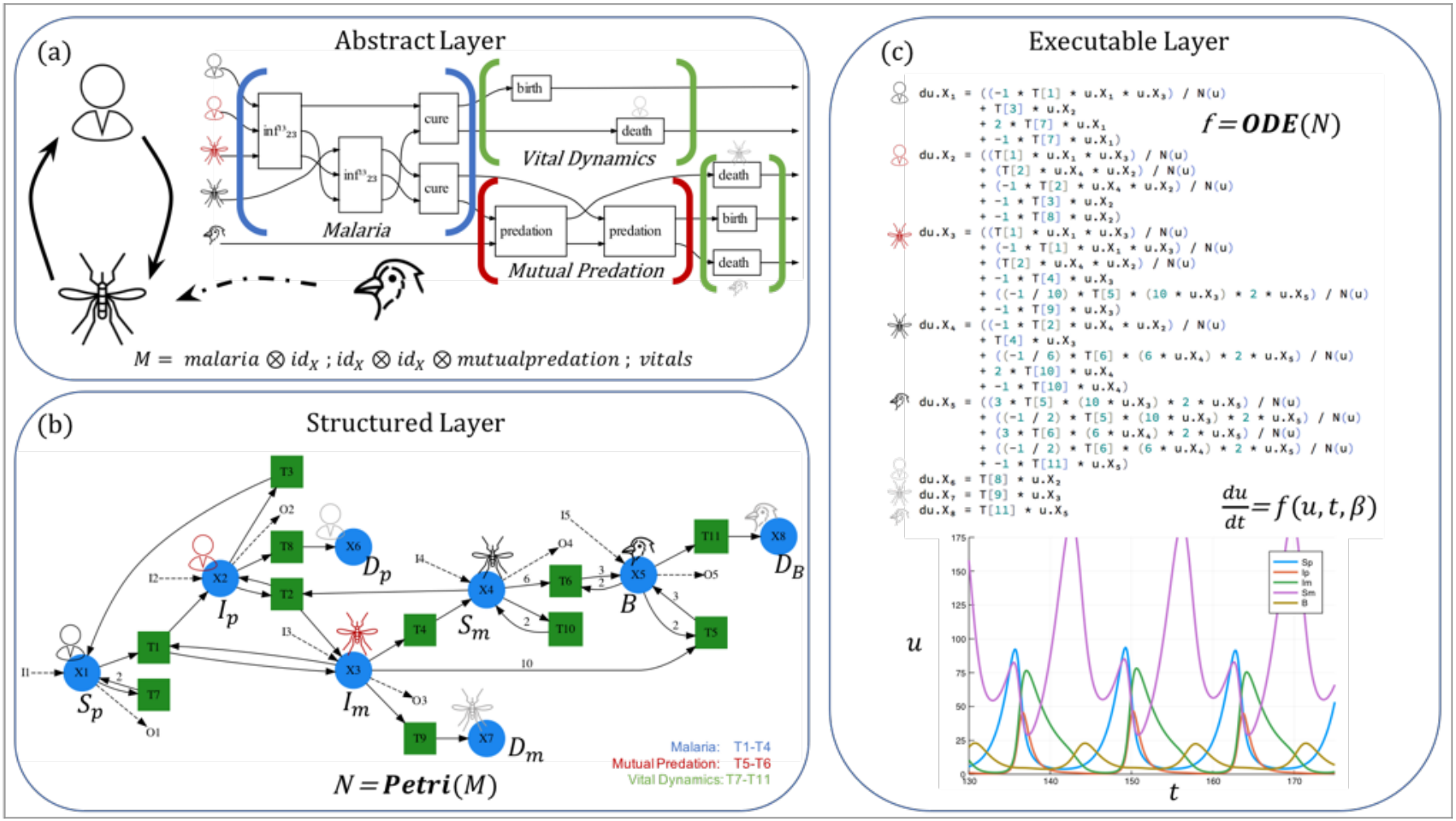
```
@present GraphSchema(FreeCategory) begin
  V::Ob
  E::Ob
  src::Hom(E,V)
  tgt::Hom(E,V)
end
Graph = CSetType(GraphSchema, index=[:src,:tgt])
```

$$\mathcal{C} = \left\{ \begin{array}{c} E \\ \mathsf{src} \downarrow\downarrow \mathsf{tgt} \\ V \end{array} \right\}$$

$$\textbf{Graph} = \textbf{Fun}(\mathcal{C}, \textbf{Set})$$

- Catlab generates efficient data structure for any schema $\mathcal{C}$
- Costless abstraction: high-level specification w/o performance penalty

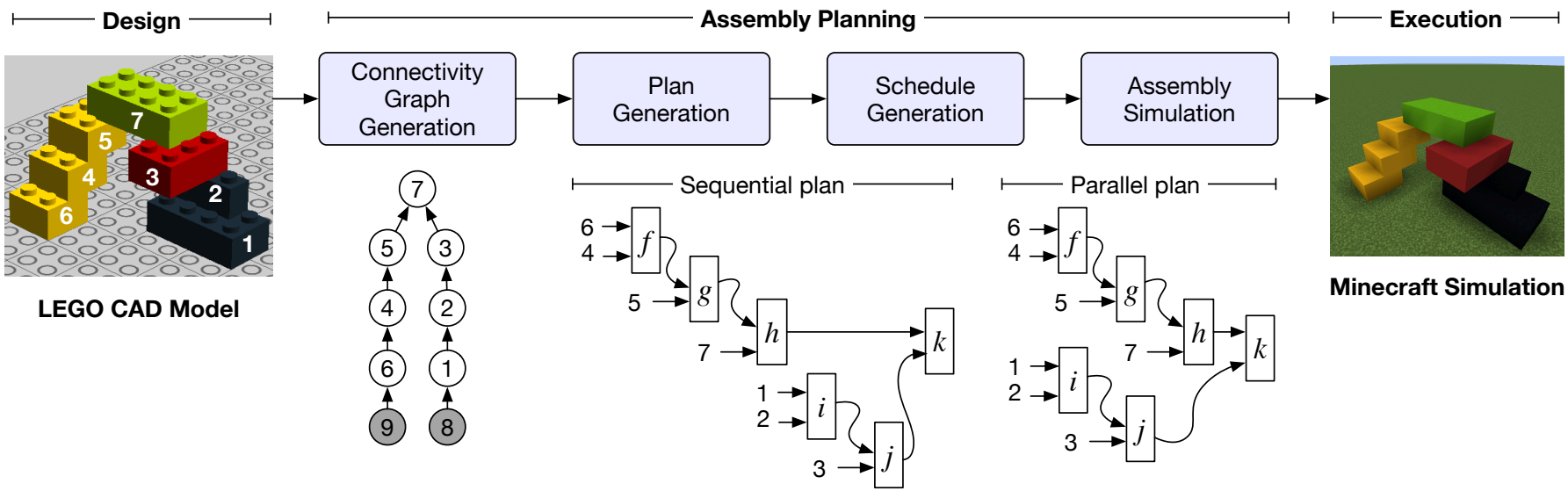
Application: Petri nets and epidemiological models



(Fairbanks, Halter, et al 2019-20: arXiv:1907.03536, arXiv:2005.04831)

Further applications

- String diagrams for assembly planning
  (Master, Patterson, et al 2019: arXiv:1909.10475)



- Semantic models of data science code
  (Patterson, Baldini, et al 2018: arXiv:1807.05691)

DoD sponsored programs

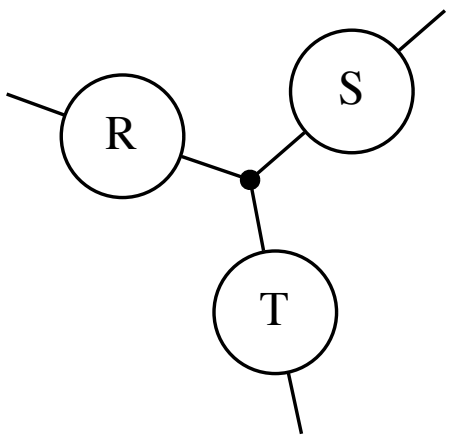
Catlab is now being used in several DoD sponsored programs:

- AFOSR: *Dynamical Systems and Control* (this program)
- ONR: *Extracting, Explaining, and Estimating Information in Sonar Data* (E3ISD)
- DARPA: *Automating Scientific Knowledge Extraction* (ASKE), Phases 1-3
- DARPA: *Artificial Social Intelligence for Successful Teams* (ASIST)
- DARPA: *Computable Models* (COMP Mods), Phase 1

Case study: Undirected wiring diagrams

Undirected wiring diagrams describe the composition of

- behavior contracts for dynamical systems
- pixel arrays (discretized continuous relations)
- tables in a SQL database, by joins
- tensor networks in quantum physics and chemistry
  - Steady states—organized by parameters and outputs—is a functor from dynamical systems networks to tensor networks.



The same category-theoretic Julia code handles all these applications.

Undirected wiring diagrams as C-sets

$$C = \left\{ P_{\text{out}} \xrightarrow{\text{junction}_{\text{out}}} J \xleftarrow{\text{junction}} P \xrightarrow{\text{box}} B \right\}$$

$$\mathbf{UWD} = \mathbf{Fun}(C, \mathbf{Set})$$

```
@present SchemaUWD(FreeCategory) begin
  Box::Ob
  Port::Ob
  OuterPort::Ob
  Junction::Ob

  box::Hom(Port,Box)
  junction::Hom(Port,Junction)
  outer_junction::Hom(OuterPort,Junction)
end
UntypedUWD = CSetType(SchemaUWD, index=[:box,:junction,:outer_junction])
```

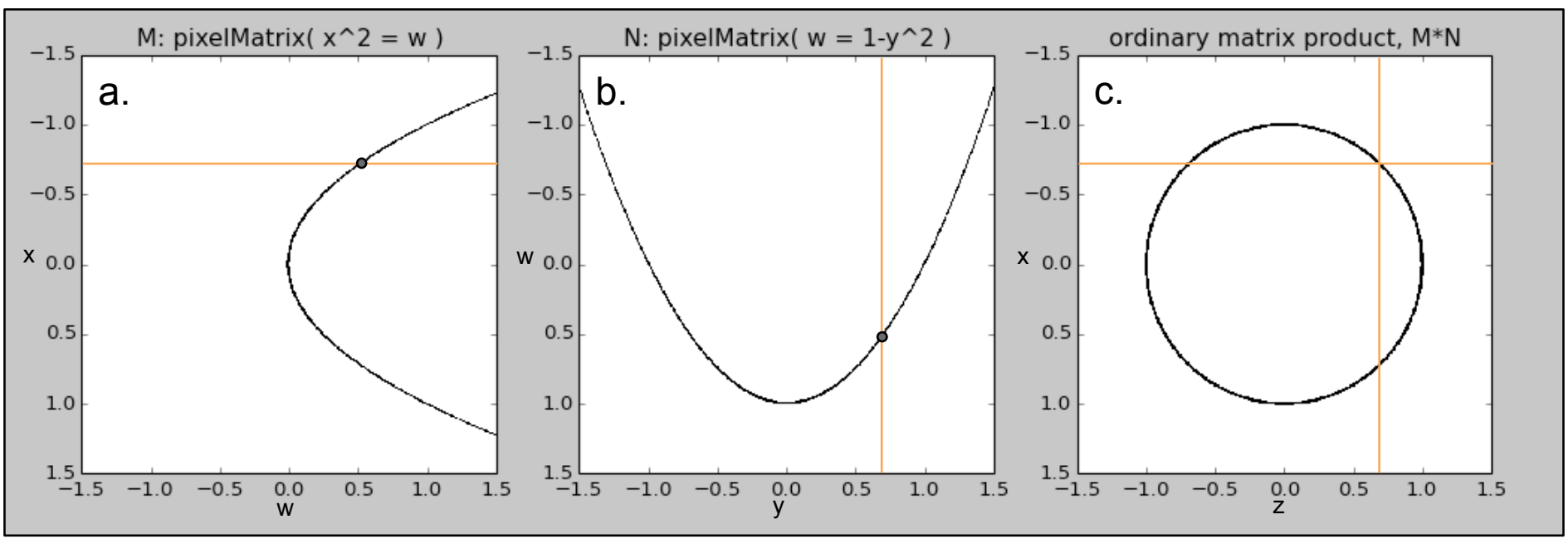
Another look at matrix multiplication

Separately plot the solutions to equations: $f(x, w) = 0$ and $g(w, y) = 0$.

- Plot each in its own bounding box, say in the range $[-1.5, 1.5]$.
- Consider the plots as matrices M, N whose entries are on/off pixels.
- That is, M and N are *boolean matrices* corresponding to f and g.

Multiplying these two matrices MN yields... the simultaneous solution.

- For example, plot equations $x^2 = w$ and $w = 1 - y^2$, and multiply.



A more complex example

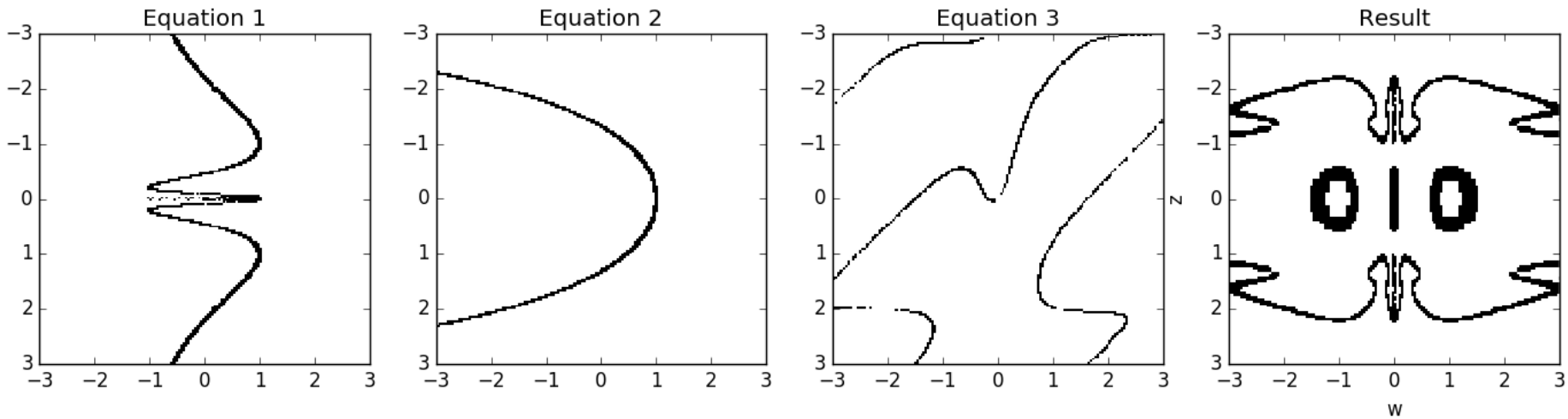
The following eq's are not differentiable, nor even defined everywhere.

$$\cos\left(\ln(z^2 + 10^{-3}x)\right) - x + 10^{-5}z^{-1} = 0 \qquad \text{(Equation 1)}$$

$$\cosh(w + 10^{-3}y) + y + 10^{-4}w - 2 = 0 \qquad \text{(Equation 2)}$$

$$\tan(x + y)(x - 2)^{-1}(x + 3)^{-1}y^{-2} - 1 = 0 \qquad \text{(Equation 3)}$$

Q: For what values of w and z does a simultaneous solution exist? [1]

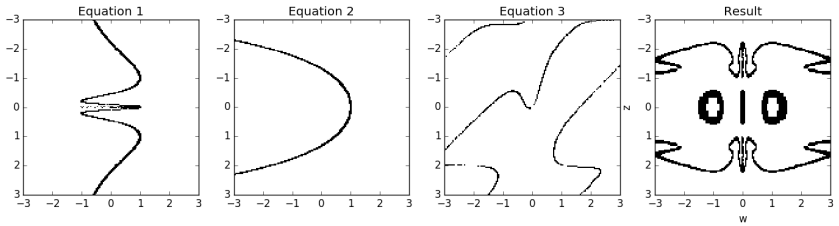


[1] Spivak, DI; Dobson, MRC; Kumari, S. (2016) "Pixel Arrays: A fast and elementary method for solving nonlinear systems". `http://arxiv.org/pdf/1609.00061v1.pdf`

Selling points

The Pixel Array method has the following features:

- it approximates *all solutions in a given bounding box*;
- it's *much faster* than quasi-Newton methods for finding "all solutions" to partially decomposable systems;
- it introduces *no false negatives*;
- it works for *non-differentiable* or even *discontinuous* functions;
- it's *not iterative* and requires no initial guess, in contrast with quasi-Newton methods;
- it's *functorial* with respect to steady states/periodic orbits of interconnected dynamical systems; and
- it *provides insights*, by showing the whole solution set at once.



Pixel arrays in Catlab

Leverage Julia ecosystem for tensors:

- Tullio.jl and other packages provide generic tensor computing
- Catlab provides `BoolRig` type for boolean-valued tensors

$$
\begin{aligned}
R_1(w,x,y) &\iff \tan(y+w) + \exp(x) = 2 \\
R_2(v,x,y) &\iff x^3 + \cos(\log(y^2)) = 1.5v \\
R_3(v,w,z) &\iff w + z + 10^{-1}v = 0.5
\end{aligned}
$$

```
@tullio R1[i,j,k] := (abs(tan(y[k] + w[i]) + exp(x[j]) - 2) < ϵ) |> BoolRig
@tullio R2[i,j,k] := (abs(x[j]^3 + cos(log(y[k]^2)) - 1.5v[i]) < ϵ) |> BoolRig
@tullio R3[i,j,k] := (abs(w[j] + z[k] + v[i]/10 - 0.5) < ϵ) |> BoolRig
```

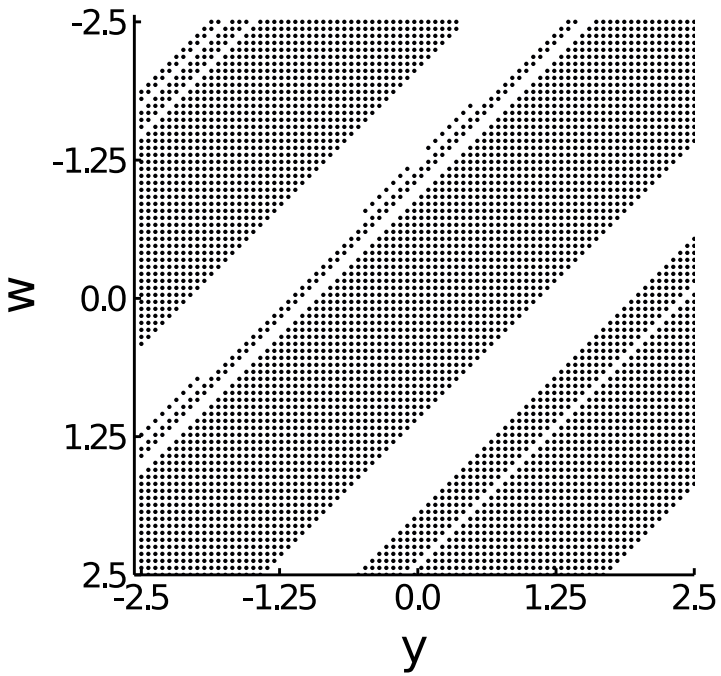
Contraction of pixel arrays

Generate Julia expression in Einstein notation:

```
compile_tensor_expr(diagram)
```

Output can be used with Tullio.jl and other packages:

```
:(out[w, y] = R1[w, x, y] * R2[v, x, y] * R3[v, w, z]) # output of above
```



Outline

Summary

We look for abstractions that will be reusable across domains.

- This lets us work with a large number of different disciplines.
- Factor hypercomplex problems; use functors to hand off fragments.

But until now, much of this has remained stuck in academic papers.

- The ACT community has talked about fascinating possibilities.
- With tooling like CQL, UMAP, and Catlab we show it actually works.
- AFOSR funding has been integral to establishing ACT, and we won't disappoint.

Thanks; comments and questions welcome!

Relevant group papers July 2019 – 2020

- Antonante; **Spivak**; Carlone (2020) "Monitoring and Diagnosability of Perception Systems". `https://arxiv.org/abs/2005.11816`
- **Fong, Spivak** (2019) *An Invitation to Applied Category Theory: Seven Sketches in Compositionality*. Cambridge University Press.
- **Fong**, Myers, **Spivak** (2020) "Behavioral Mereology: A Modal Logic for Passing Constraints", *Proceedings of the 3rd International Conference on Applied Category Theory*.
- **Fong**, Sarazola (2020) "A recipe for black box functors" *Theory and Applications of Categories* Vol. 35, No. 26, pp 979-1011.
- **Fong, Spivak, Tuyéras** (2019) "Backprop as Functor: A compositional perspective on supervised learning". *LICS*.
- Fritz, **Perrone**, Rischel, Gonda (2020) "Stochastic dominance and the Blackwell theorem in Markov categories." *To appear*
- Genovese, F.; **Spivak** (2020) "A Categorical Semantics for Guarded Petri Nets". *ICGT*.
- Patterson, E.; **Spivak**; Vagner, D. (2020) "Wiring diagrams as normal forms for computing in symmetric monoidal categories". *Proceedings of the 3rd International Conference on Applied Category Theory*.
- **Spivak** (2020) "Poly: An abundant categorical setting for mode-dependent dynamics."
- Zardini G.; **Spivak**; Censi, A; Frazzoli, E. (2020) "A Compositional Sheaf-Theoretic Framework for Event-Based Systems". *Proceedings of the 3rd International Conference on Applied Category Theory*.