The free monad, cofree comonad, and topological space associated to a polynomial

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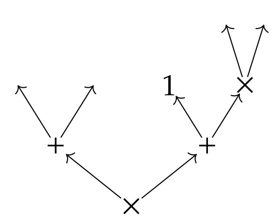
Abstract

To each polynomial p, one can associate a free monad $\mathfrak{m}_p$, the "operad of well-founded p-trees", and a cofree comonad $\mathfrak{c}_p$, the "category of (possibly infinite) p-trees". It turns out that the object-set $\mathfrak{c}_p(1)$ of the latter category forms the point-set of a topological space, which is called a Stone space when p is finite, and that the morphism-set $\mathfrak{m}_p(1)$ of the operad forms a base for this topology! I imagine the topological space $(\mathfrak{c}_p(1), \mathfrak{m}_p(1))$ could provide an alternative point of view for studying p-dynamical systems, or more precisely p-coalgebras. I'll explain what I know about all this, starting from very little background beyond standard category theory.

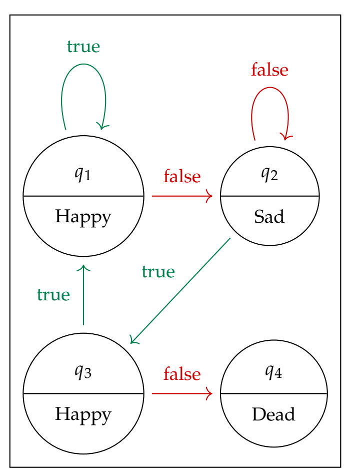
I. Introduction

A. **Poly** as "the true language of computation": dependent types, state machines, and databases — 1 min

B. What's the relationship between "language" (monad) and "experience" (comonad) — 3 mins



$(x_1 + x_2) \times (1 + (x_3 \times x_4))$



C. Two toposes associated with any polynomial p: 2 mins

1. p-**Coalg** $\cong \mathfrak{c}_p$-**Set**, "singular experience"
2. $\mathbf{Shv}(\mathcal{S}_p)$, for a certain space $\mathcal{S}_p$, "counterfactual plans"

D. Plan for talk 1 min

1. Background
2. Comonads, coalgebras, monads, algebras
3. Topological space with points $\mathfrak{c}_p(1)$, basic opens $\mathfrak{m}_p(1)$
4. Finding the monad map $\mathfrak{m}_p \to [\mathfrak{c}_p, \mathfrak{m}_{p+1}]$

II. Background on $(\mathbf{Poly}, 0, +, 1, \times, y, \otimes, \triangleleft)$

A. Representable functors $y^A \coloneqq \mathbf{Set}(A, -)$

B. **Poly** = sums of representables, $p = \sum_{I \in p(1)} y^{p[I]}$

C. Coproduct and product as usual $(0, +)$, $(1, \times)$.

D. Dirichlet product $(y, \otimes)$

E. Composition product $(y, \triangleleft)$

1. $p \triangleleft q$ as p-corollas under q-corollas.
2. $p^{\triangleleft N}$

III. Coalgebras and comonads, algebras and monads

A. p-coalgebras as dynamical systems 1 min

B. $\mathfrak{c}_p = \lim(\cdots \to y(p \triangleleft y(p \triangleleft 1)) \to y(p \triangleleft 1) \to 1)$ 4 mins

C. Theorem: 4 mins

1. p-**Coalg** $\cong \mathfrak{c}_p$-**Set**,
2. $\mathfrak{c}_p$ is cofree,
3. $\mathfrak{c}_p \cong y(p \triangleleft \mathfrak{c}_p)$
4. Terminal coalgebra is $\mathfrak{c}_p(1)$

D. p-algebras as models for syntax 1 min

E. $\mathfrak{m}_p = \operatorname{colim}(\cdots \leftarrow y + p \triangleleft (y + p \triangleleft 0) \leftarrow y + p \triangleleft 0 \leftarrow 0)$ 4 mins

F. Theorem: Assume p is finitary 4 mins

1. p-$\mathbf{Alg}_{\text{Lambek}} \cong \mathfrak{m}_p$-$\mathbf{Alg}_{\text{E-M}}$
2. $\mathfrak{m}_p$ is free
3. $\mathfrak{m}_p \cong y + p \triangleleft \mathfrak{m}_p$
4. Initial p-algebra is $\mathfrak{m}_p(0)$
5. For any set X, bijection: $\mathfrak{m}_p(X) \cong \mathfrak{m}_{p+X}(0)$

G. Consequences of Lambek's theorem — 3 mins

1. Lambek: $\mathfrak{c}_p(1) \cong p \triangleleft \mathfrak{c}_p(1)$ and $p \triangleleft \mathfrak{m}_p(0) \cong \mathfrak{m}_p(0)$
2. So we have $\mathfrak{m}_p \triangleleft \mathfrak{c}_p(1) \to \mathfrak{c}_p(1)$; picture it.
3. And we have $\mathfrak{m}_p(0) \to \mathfrak{c}_p \triangleleft \mathfrak{m}_p(0)$; picture it.

IV. Two toposes associated to p — 4 mins

A. The topos p-**Coalg** $\cong \mathfrak{c}_p$-**Set**.

1. Logic is "whenever statements"
2. Can't talk about this moment, only fate
3. Experience always follows one path up the tree

B. Sheaves on a space $\mathfrak{S}_p$ — 8 mins

1. The points of $\mathfrak{S}_p$ is $\mathfrak{c}_p(1)$
2. Topology is the prodiscrete topology, i.e. the limit

$$\mathfrak{S}_p = \lim(\cdots \to p^{\triangleleft N}(1) \to \cdots \to p^{\triangleleft 2}(1) \to p(1) \to 1)$$

3. If p finite, each $p^{\triangleleft N}(1)$ is finite; $\mathfrak{S}_p$ is "profinite space"
4. Theorem: TFAE
   a. Profinite spaces
   b. Stone spaces: compact, Hausdorff, totally disconnected
   c. (Boolean algebras)$^{\text{op}}$
5. The logic can't talk about moving through time, only finite-time counterfactuals starting now

V. $\mathfrak{m}_p(1)$ as base for topology on $\mathfrak{c}_p(1)$.

A. Open sets of $\mathfrak{S}_p$ — 4 mins

1. The set of trees that truncate to a given finite tree is open
2. All opens are unions of such
3. But in fact $\mathfrak{m}_p(1)$ also forms a basis

B. We can obtain the relation $\mathfrak{m}_p(1) \to [\mathfrak{c}_p(1), 2]$ via a monad map 1 min

$$\text{Gendlin}\colon \mathfrak{m}_p \to [\mathfrak{c}_p, \mathfrak{m}_{p+1}]$$

C. Let's derive the relation from the Gendlin map 3 mins

1. For any p, we have an p-algebra structure $p \triangleleft 2 \to 2$ by

$$(I, b) \mapsto \bigwedge_{i \in p[I]} b_i$$

   where $I \in p(1)$ and $b\colon p[I] \to \{\text{false,true}\}$.
2. Given g, get $\mathfrak{m}_p(1) \to [\mathfrak{c}_p(1), \mathfrak{m}_{p+1}(1)]$ by duoidality
3. Recall $\mathfrak{m}_{p+1}(1) \cong \mathfrak{m}_p(2)$
4. Use p-algebra and hence $\mathfrak{m}_p$-algebra structure on 2 to get

$$\mathfrak{m}_p(1) \to [\mathfrak{c}_p(1), \mathfrak{m}_{p+1}(1)] \to [\mathfrak{c}_p(1), \mathfrak{m}_p(2)] \to [\mathfrak{c}_p(1), 2]$$

D. Gendlin map $\mathfrak{m}_p \to [\mathfrak{c}_p, \mathfrak{m}_{p+\{\text{`false`}\}}]$ 4 mins

1. Idea: for any finite tree, assign to each infinite tree the subtree where they agree, saying "false" when they don't
2. We'll use `unfold`: $\mathfrak{m}_p \to \{\text{`true`}\}y + p \triangleleft \mathfrak{m}_p$.
3. Algorithm for @: $\mathfrak{m}_p \otimes \mathfrak{c}_p \to \mathfrak{m}_{p+1}$

```
u @ t := case (unfold u) of
  ‘true‘  => (‘true‘, () |-> ((), id))
  (I, u’) => if (t.root == I)
    then (I, i |-> (u’.i @ t.i))
    else (‘false‘, absurd)
```