Dynamic operads: Compositional adaptive systems

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Outline

Categorical systems theory

I've been funded by DoD since 2008 to develop *Applied Category Theory*.

- CT is the mathematics of compositional structures and relationships.
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- ACT is growing: since 2018, a journal, int'l conf., research institute.

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- Structure-preserving maps and operations let you probe and build.
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Today: discuss categorical systems theory. What *structure* is prevalent?

Compositional adaptive systems

A system is a collection of interacting parts that form an integrated whole.

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- In the nervous system, a collection of nerve cells interact.

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- We also see it in living systems: they adapt to feedback.
- What is the compositional structure underlying all these stories?

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Rather than "complex" adaptive systems: *compositional* ones.

- Baking complexity into the name precludes studying simple systems.
- It'd be like studying $\mathbb{N} - \{0, \cdots, 100\}$. It's good to include 0.
- We see compositionality everywhere. Let's bake that in instead.

Plan for the talk

For the remainder of the time I will:

- Explain key notions: interfaces and dynamic arrangements;
- Hint at mathematical definition: dynamic operads;
- Show examples: neural nets and compositional classical mechanics;
- Conclude.

Outline

How are we conceptualizing systems

I want a formalism that bakes in compositionality.

- We often see systems of systems of systems.
- The behavior of the whole is not determined by that of the parts, but...
- ...it is determined by the behavior of the parts and *how they interact.*
- New collectively-reinforcing patterns can *emerge* as the parts interact.
- What emerges becomes basic behavior at the next compositional level.

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- They interact with other systems through some sort of *interface*.
- This is offers a low-dimensional projection of their state space.
- The interface not only describes how information leaves the system,...
- ...but also how information enters the system, parameterizing behavior.
- What can the system express, and what can impress upon the system?

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What's a broad working definition for the general structure of interfaces?

Mathematics of interfaces

For this talk, we will formalize interfaces as “bundles” in a general sense.

- Pick a category $\mathcal{M}$ with products, e.g. sets or smooth manifolds.
- Pick a notion of bundle $\mathrm{V}\colon \mathcal{M}^{\mathsf{op}} \to \mathbb{C}\mathbf{at}$.
- E.g. for V you could use “vector bundles on manifolds” $E \to B$.
- Or you could use submersions of manifolds or maybe maps in $\mathcal{M}$.

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The theory is pretty agnostic about how you choose $\mathcal{C}, \mathrm{V}$.

- Rule 0: given $f\colon M' \to M$, need "pullback" map $f^*\colon \mathrm{V}(M) \to \mathrm{V}(M')$
- Rule 1: need $\mathrm{V}(M_1) \times \mathrm{V}(M_2) \xrightarrow{\otimes} \mathrm{V}(M_1 \times M_2)$ ...i.e. V lax monoidal.
- An *interface* is just a pair (M, V) with $M : \mathcal{M}$ and $V : \mathrm{V}(M)$.

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This interface tells us how information will go to and from the system.

- Suppose a system is housed in an interface (M, V).
- Think of M as all the outputs, the ways the system expresses its state.
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Example: $\mathcal{M} = \mathbf{Mfd}$, $\mathrm{V}(M) =$"vector bundles on M".

- Example interfaces: $\binom{T^*M}{M}$ and $\binom{TM}{M}$, (co)tangent bundles.
- The system outputs points $m : M$ and inputs (co)tangent vectors at m.

Arrangements 1 of 3

Unless o'wise specified, we're agnostic to $\mathcal{M}, \mathrm{V}$, but we'll talk in manifolds.

- An *interface* is a pair $\binom{V}{M}$ where M : **Mfd** and $V \to M$ is a v.bundle
- Output space is M; given $m : M$, input space is fiber $V_m \coloneqq m^*(V)$.
- E.g. for $V = \mathbb{R}^m \times \mathbb{R}^n$, we're outputting $x : \mathbb{R}^m$ and inputting $y : \mathbb{R}^n$.
- For arb. M : **Mfd** and $V \coloneqq T^*M$, output m, input covector $x : T^*_m M$.
- Later we will put open dynamical systems in these interfaces $\binom{V}{M}$.

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We will consider static arrangements now; later: dynamic arrangements.

- We'll make a system of systems by arranging interfaces in an interface.
- That is, we'll define a notion of map $\binom{V_1}{M_1} \otimes \cdots \otimes \binom{V_k}{M_k} \leftrightarrows \binom{W}{N}$.
- First, let $\binom{V_1}{M_1} \otimes \binom{V_2}{M_2} := \binom{V_1 \times V_2}{M_1 \times M_2}$, e.g. $\binom{TM}{M} \otimes \binom{TN}{N} \cong \binom{T(M \times N)}{M \times N}$.

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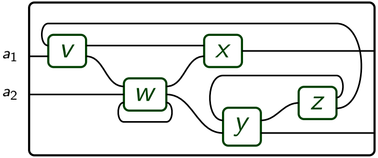
So what do we take to be a map $\binom{V}{M} \leftrightarrows \binom{W}{N}$?

- Define it to be a pair $(f, f^\sharp)$ where $f \colon M \to N$ smooth and...
- ...$f^\sharp \colon f^*W \to V$. That is, pullback W to M and then map to V.
- Next we'll explain what this means and give an example.
- Then we'll discuss dynamic arrangements.

Arrangements 2 of 3

How should we think of an arrangement $\binom{V_1}{M_1} \otimes \cdots \otimes \binom{V_k}{M_k} \leftrightarrows \binom{W}{N}$?



The above is an arrangement of type $\binom{\mathbb{R}^2}{\mathbb{R}^2}\otimes\binom{\mathbb{R}^3}{\mathbb{R}^3}\otimes\binom{\mathbb{R}^2}{\mathbb{R}^1}\otimes\binom{\mathbb{R}^2}{\mathbb{R}^2}\otimes\binom{\mathbb{R}^1}{\mathbb{R}^2} \leftrightarrows \binom{\mathbb{R}^2}{\mathbb{R}^2}$

- Here, we write $\binom{\mathbb{R}^n}{\mathbb{R}^m}$ to denote the trivial bundle on $\mathbb{R}^m$ with fiber $\mathbb{R}^n$.
- The map comes in two pieces: first, $f\colon \mathbb{R}^2\times\mathbb{R}^3\times\mathbb{R}^1\times\mathbb{R}^2\times\mathbb{R}^2 \to \mathbb{R}^2$.
- Check the diagram: $f(v_1, v_2, w_1, w_2, w_3, x_1, y_1, y_2, z_1, z_2) \coloneqq (x_1, y_2)$.
- Second, a map $f^\sharp\colon f^*(\mathbb{R}^2) \to \mathbb{R}^2\times\mathbb{R}^3\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^1$.
- Check the diagram: $f^\sharp(v_1, v_2, w_1, w_2, w_3, x_1, y_1, y_2, z_1, z_2, a_1, a_2) \coloneqq (z_2, a_1, v_2, a_2, w_3, v_1, w_1, z_1, w_2, y_1)$
- Inner systems trade information between themselves and outside world.

We'll gener'ze in two ways: non-trivial bundles and dynamic arrangements.

Arrangements 3 of 3

Even static arrangements are much more general than wiring diagrams.

- Any map of manifolds $f\colon M_1 \times \cdots \times M_k \to N$ induces
  $$\binom{T^*M_1}{M_1} \otimes \cdots \otimes \binom{T^*M_k}{M_k} \leftrightarrows \binom{T^*N}{N}$$
- Reason: given f, we get $T^*f\colon T^*N \to T^*(M_1) \times \cdots \times T^*(M_k)$.

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But we want dynamic arrangements: arrangements that can change.

- Want that what flows through the system can change the arrangement.
- Companies: communication among people can change the structure.
- Cyberphysical systems: signals can change the way energy flows.
- Deep learning: training data can change the weights.
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Let's consider the running example of manifolds and vector bundles.

- Define a dynamic arrangment from $\binom{V}{M}$ to $\binom{W}{N}$ to consist of:
- ...a choice of manifold X and a map $\binom{V}{M} \otimes \binom{TX}{X} \leftrightarrows \binom{W}{N}$.
- Given a curve in M and a field $N \to W$, get time-varying v.field on X.
- Why call this dynamic? Ans. 1: if $X = \mathbb{R}^0$ recover static arrangement.
- Ans. 2: The arrangement is param'd by $x : X$, and this changes.

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Dynamic operads

Now we have interfaces and arrangements; how do they compose?

- A (typed) operad $\mathcal{O}$ is an algebraic structure for multi-ary operations.
- They come up in logic, programming, tensor networks, grammars, etc.
- To define an $\mathcal{O}$: specify its types, operations, and composition rules.

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When $\mathcal{O}$'s operations form a "blah", it's called a "blah"-enriched operad.

- Unadorned, assume $\mathcal{O}$ has a *set* of operations, so it's **Set**-enriched.
- But $\mathcal{O}$ could have operations that vector spaces.
- For any mon'l cat'y $\mathscr{D}$ (e.g., vector spaces), have $\mathscr{D}$-enriched operads.

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In our case, we will be interested in dynamically-enriched operads.

- Rather than a set of operations, we have a dynamical system...
- ...each of whose states forms an operation.
- What we've been calling "arrangements" are the operations.
- Dynamic arrangements $\binom{V}{M} \leftrightarrows \binom{W}{N}$ change in time.
- That is, there is a time-action on the space of operations.

Examples

We've proven that this sort of structure comes up in diverse applications.

- One must prove many coherences to ensure things "work as expected".
- CT suggests the right definitions, and they hold in our examples.
- Examples: neural networks, Hebbian learning, prediction markets.

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The neurons in a neural network form a dynamic operad.

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- Dynamic arrangements $\binom{T^*\mathbb{R}^{m_1}}{\mathbb{R}^{m_1}} \otimes \cdots \otimes \binom{T^*\mathbb{R}^{m_k}}{\mathbb{R}^{m_k}} \otimes \binom{T\mathbb{R}^d}{\mathbb{R}^d} \leftrightarrows \binom{T^*\mathbb{R}^n}{\mathbb{R}^n}$.
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Where does this come from? Often use $d := (m_1 + \cdots + m_k) \times n$.

- Given a map $\mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_k} \times \mathbb{R}^d \to \mathbb{R}^n$, e.g. "activated matmul",...
- ...one obtains a map backwards on cotangent vectors.
- Finish by using the canonical Riemannian structure $T^*\mathbb{R}^d \xrightarrow{\cong} T\mathbb{R}^d$.
- Generalize by replacing $\mathbb{R}^{m_i}$ and $\mathbb{R}^n$ with arb. manifolds and $\mathbb{R}^d$...
- ...with any X : **Mfd** having a "feedback response" map $T^*X \to TX$.
- Example: Riemmannian and symplectic mfds X have such structures.

Compositional physics

What makes artificial networks work?

- It's unknown why they're so good at their job.
- But what makes them feasibly fast is their compositional structure.
- Gradient back-propagation is compositional by the chain rule.
- But gradient *descent* is extra! It's comes from Riemann $T^*X \to TX$.
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Compositional physics is formally analogous.

- Hamiltonian dynamics replaces gradient descent dynamics.
- I'm not a physicist nor geometer; joint with Owen Lynch.
- Think of network architecture as a causal factoring of your system.
- In a dynamic arrangement $\binom{T^*M}{M} \otimes \binom{TX}{X} \leftrightarrows \binom{T^*N}{N}$,...
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Example: a system of colliding balls, each with springs or balls inside.

- The compositional physics allows computation by message passing.
- Forces on boundary are transferred inside, momentum is output.
- Physics sim'n & ANN training can be combined as one procedure.

Putting things together

Compositional adaptive systems in nature.

- Prediction markets, Hebbian learning, and ANNs have a similar flavor.
- We can see the system "adapting" to feedback.
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- The current arrangement dictates the input-output behavior, yet...
- ...what goes on throughout the system changes this arrangement.
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One can thus combine neural nets and Hamiltonian physics engines.

- Backprop in ANNs is formally analogous to force transfer via b'ndary.
- One software architecture includes both simulation and learning.
- Structural thinking lets us find conceptual neighbors, generic theorems.

Outline

Summary

We want a very general approach to systems that adapt to feedback.

- Category theory is all structure, agnostic as to content.
- It helps you articulate the types and how they relate.
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We articulated the structure of interfaces and dynamic arrangements.

- Very general setup: lax monoidal $\mathrm{V}\colon \mathcal{M}^{\mathsf{op}} \to \mathbf{Cat}$.
- Interfaces are pairs (X, V), denoted $\binom{V}{X}$, where $X : \mathcal{M}$ and $V : \mathrm{V}(X)$.
- Example: $\binom{T^*X}{X}$. Output an element of X, receive a cotangent vector.
- Arrangements of little systems in a big system define inform'n flow.
- Arrangements are dynamic: they change based on what flows.

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- Any feedback response $T^*X \to TX$ gives a sys'matic way of changing.
- Content-agnostic structure let's us see big-picture commonalities.

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Thanks; comments and questions welcome!