

Applied Category Theory: Towards a hard science of interdisciplinarity

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Chevron Systems Engineering Community of Practice
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Outline

1 Introduction

- Why am I here?
- Accounting for interdisciplinarity
- Plan for the talk

2 Operads: a framework for compositional operations

3 Dynamic operads

4 Conclusion

Why am I here?

In 2007, I read *The Moment of Complexity* by Mark C. Taylor

- It explained that the world was getting increasingly complex.
- More would be different: Anomalies would become the norm.
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I wanted to do something, to help the world navigate. But how?

- Each person and organization structures its experience of the world.
- My neuron pattern and yours are more different than our fingerprints.
- And these are quite different than the database schemas found in orgs.
- Despite these massive differences, we are able to communicate! How?
- For the world to navigate, we must communicate better.
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Navigation requires coordination. Our efforts need to be more coherent.

- We need to understand interdisciplinarity: how to learn from others.
- I want this to be a hard science, i.e. to be supported by math.

Accounting

We solve big problems together by coordinating our activity.

- When my efforts and yours conflict, it causes friction and loss.
- When we coordinate, we stop stepping on each others' toes.
- To work collectively, our activities must align. How do we align them?
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Note: **regularity** is different than predictability.

- A chess game is **regular** (pawns don't move left), not predictable.
- Regulation: "Hey, you can't move a pawn left"; "Oh, oops!"

Mathematical fields as accounting systems

I think of mathematical fields as **crystalized accounting systems**.

- Arithmetic accounts for the flow of quantities, as in finance.
- Hilbert spaces account for the states of elementary particles, as in QM.
- Probability distributions account for likelihoods, as in game theory.
- Calculus accounts for relative rates of change.
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- Carefully **track** the phenomena, **articulate** the structure, **systematize**.
- So we want to track and articulate the structure of sense-making.

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Category theory (CT) is the accounting system for interlocking structures.

- Mathematical definitions are composed of interlocking structures.
- Category theory tracks the layers of structure and their connections.
- This makes analogies—similarities of structure—into formal objects.
- It accounts for the fact that different accounting systems cohere.

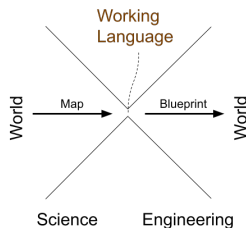
Science, math, and engineering

Roughly: science compresses, engin'ring elaborates.

- Scientists map of the world: lift out patterns.
- Engineers make blueprints to reproduce them.

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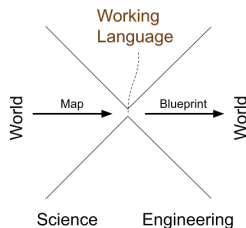
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What would a hard science of interdisciplinarity be?

- Hard sciences are ones that are more mathematically accountable.
- We need mathematical accounting to scaffold interdisciplinarity itself.
- I.e. math that accounts for working analogies between disciplines.



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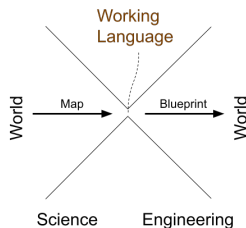
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Category theory is like a conceptual stem cell.

- A stem cell can differentiate into huge variety of forms.
- Coming from a common origin, these forms work together coherently.



Category theory as conceptual stem-cell

Category theory (CT) can differentiate into many forms:

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Category theory (CT) can differentiate into many forms:

- All forms of pure math... (algebra, topology, logic, number theory, differential equations...)
- Databases and knowledge representation (categories and functors)
- Functional programming languages (cartesian closed categories)
- Dynamical systems and fractals (operad-algebras, co-algebras)
- Shannon Entropy (operad of simplices, internal algebras)
- Taxonomies, metric spaces, and networks (enriched categories)
- Measurements of diversity in populations (magnitude of categories)
- Open economic game theory (Lens categories)
- Collaborative design (enriched categories and profunctors)
- Petri nets and chemical reaction networks (monoidal categories)
- Quantum processes and NLP (compact closed categories)
- Disease modeling and compartmental models (hypergraph categories)
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Popper's objection

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We counter this objection in two ways:

- Couldn't the same objection be made about mathematics?
 - Mathematics is the basis of hard science, used everywhere.
 - CT—like math—explains, models, formalizes many many things.
 - Conclude that math/CT explains everything and hence nothing?

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 - Conclude that math/CT explains everything and hence nothing?
- Stem cells don't do work until they differentiate.
 - “Adult-level” work requires differentiation and optimization.
 - But the unified origins lead to impressive interoperability.
 - That's what we need for interdisciplinarity.

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- And it's branched out from math in a big way.
 - Databases and knowledge representation ([categories and functors](#))
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- Operads:

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Let's focus on one.

- Operads: compositional arrangements of things within things.
- They're a branch of category theory that well-represents the spirit.

Plan of the talk

Overarching idea: CT as math to scaffold accounts from many disciplines.

- To pick one, we'll discuss *operads*: compositional arrangements.
- I'll sketch a definition and give a lot of examples.
- I'll explain dynamic operads, e.g. for learning and prediction markets.
- I'll conclude with a summary.

Outline

1 Introduction

2 Operads: a framework for compositional operations

- Operads: e pluribus unum
- Examples of operads
- Summary on operads

3 Dynamic operads

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Operations take **arrangements** of many **sorts** and produce one **sort**.

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An operad consists of:

- A collection of **sorts** $X, Y, \dots$,
- And ways to **arrange** them, $\varphi: X_1, \dots, X_k \rightarrow Y$,
- Such that arrangements can be **nested** inside each other.

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Slightly more formal definition to come.

Operads are everywhere

Operads are used unconsciously in many fields.

- Electrical engineering: “wiring diagrams”
- Design: “set-based design”
- Computer programming: “data flow”
- Natural language processing: “grammars”
- Materials science: “hierarchical materials”
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We want to bring operads to the fore.

- There's a common theme in the way we think.
- Operads structure this sort of thinking.
- With mathematical structure, we can go much further.

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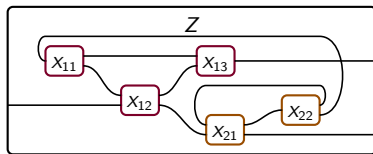
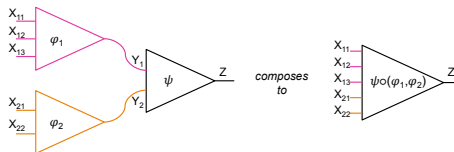
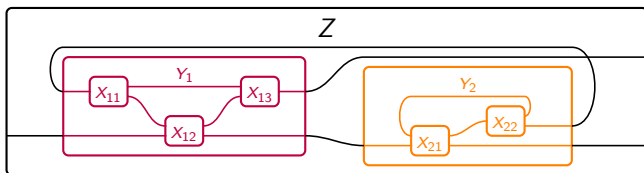
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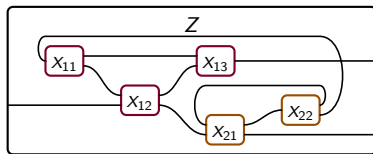
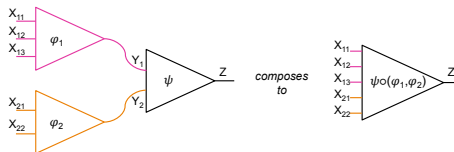
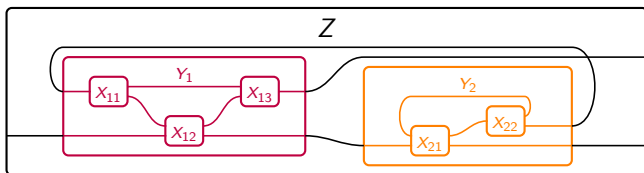
Let's look for **sorts**, **arrangements**, and **nesting** in some examples.

Operad 1: wiring diagrams



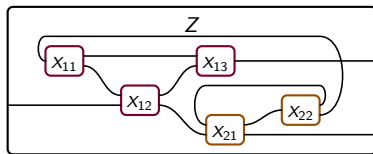
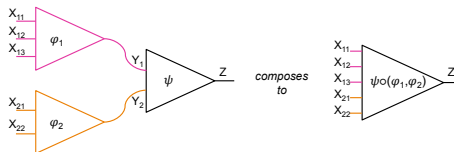
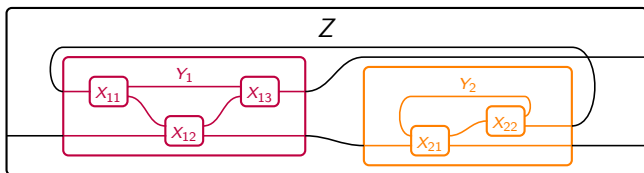
Sorts: boxes with ports.

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Sorts: boxes with ports. **Arrangements:** wiring diagrams.

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Sorts: boxes with ports. **Arrangements:** wiring diagrams. **Nesting:** nesting.

Formal definition of operad

An operad $\mathcal{O}$ consists of

- A set $\text{Ob}(\mathcal{O})$, elements of which are called *sorts*.
- For sorts $X_1, \dots, X_k, Y \in \text{Ob}(\mathcal{O})$, a set

$$\text{Mor}_{\mathcal{O}}(X_1, \dots, X_k; Y)$$

Its elements are called *morphisms* or *arrangements* of $X_1, \dots, X_k$ in Y .
A k -ary arrangement $\varphi \in \text{Mor}_{\mathcal{O}}(X_1, \dots, X_k; Y)$ may be denoted

$$\varphi: (X_1, \dots, X_k) \rightarrow Y.$$

- For each sort $X \in \text{Ob}(\mathcal{O})$, an identity arrangement $\text{id}_X: (X) \rightarrow X$.
- A composition, or *nesting* formula, e.g.,

$$\psi \circ (\varphi_1, \dots, \varphi_k): (X_{i,j}) \xrightarrow{\varphi_i} (Y_i) \xrightarrow{\psi} Z.$$

These are required to satisfy well-known “unital” and “associative” laws.

Operad 1: WDs again

An operad $\mathcal{W}$ for composing wiring diagrams:

- Sort $X \in \mathcal{W}$: any possible **box-with-ports**.



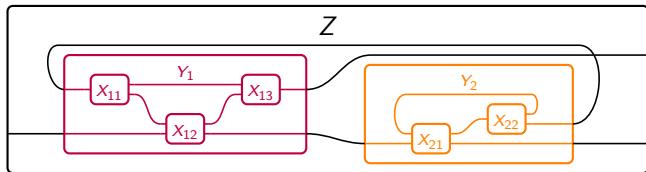
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- **Sort** $X \in \mathcal{W}$: any possible **box-with-ports**.



- **Arrangement** $\varphi: X_1, \dots, X_k \rightarrow Y$ in $\mathcal{W}$: any **wiring** of X 's in Y .
- **Nesting**: the facts about this **fractal** of wiring possibilities.



- (You could imagine an open dynamical system in each box.)

$\mathcal{W}$ is the decision of what sorts and arrangements you're considering.

Operad 2: hierarchical protein materials

There is an operad $\mathcal{M}$ for composing hierarchical protein materials.

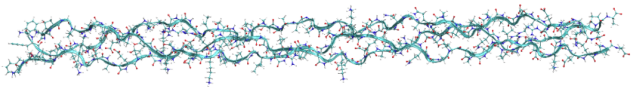
- Why protein materials?
 - Protein materials include your skin: stretchable, breathable, waterproof.
 - Eat hamburgers, make amazing material.
 - Materials scientists would *love* to make materials like this.

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 - Materials scientists would *love* to make materials like this.
- A **protein** is an **arrangement** of simpler **proteins**.
 - There are “atomic” proteins: amino acids.
 - arrange in series or parallel (H-bonds), or
 - arrange in helices, double helices, any conceivable curve, etc.



- Collagen has a **nested** structure: it is an array, each fiber of which is a triple helix, each strand of which is a helix, each unit of which is an amino acid.¹

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Operad 3: probabilities

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Arrangement: “In this event, there’s a distribution on next events.”

- coin flip: $f = (\frac{1}{2}, \frac{1}{2}) \in \mathcal{P}_2$.
 - In the event coin flip, there’s a 50-50 distribution on next events.
- die roll: $r = (\frac{1}{6}, \dots, \frac{1}{6}) \in \mathcal{P}_6$.
- card selection: $p = (\frac{1}{52}, \dots, \frac{1}{52}) \in \mathcal{P}_{52}$.

The **nesting** rule composes distributions by weighted sum:

- Flip a coin: result decides whether to roll a die or pick a card.

$$f \circ (r, p) = \left(\underbrace{\frac{1}{12}, \dots, \frac{1}{12}}_{6 \text{ times}}, \underbrace{\frac{1}{104}, \dots, \frac{1}{104}}_{52 \text{ times}} \right) \in \mathcal{P}_{58}$$

A zoo of operads: Grammars

Any context-free grammar is an operad.

$\langle \text{sentence} \rangle$	$::=$	$\langle \text{noun-phrase} \rangle \langle \text{verb-phrase} \rangle$
$\langle \text{noun-phrase} \rangle$	$::=$	$\langle \text{pronoun} \rangle \mid \langle \text{proper-noun} \rangle \mid \langle \text{determiner} \rangle \langle \text{nominal} \rangle$
$\langle \text{nominal} \rangle$	$::=$	$\langle \text{noun} \rangle \mid \langle \text{nominal} \rangle \langle \text{noun} \rangle$
$\langle \text{verb-phrase} \rangle$	$::=$	$\langle \text{verb} \rangle \mid \langle \text{verb} \rangle \langle \text{noun-phrase} \rangle \mid \langle \text{verb} \rangle \langle \text{prep-phrase} \rangle$
$\langle \text{prep-phrase} \rangle$	$::=$	$\langle \text{preposition} \rangle \langle \text{noun-phrase} \rangle$

How is this an operad?

- The **sorts** are the parts of speech.
- The **arrangements** are the production rules.
- **Nesting** is nesting.

An arboretum of operads: Recipes

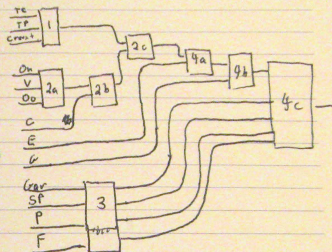
Each recipe is an operad.

- Combine sub-recipes to make a recipe.
- The outline for this talk is a recipe for getting an idea across.
 - **Sorts**: points I want to make.
 - **Arrangements**: putting points together to make a bigger point.
 - **Nesting**: the well-known outline structure.

An arboretum of operads: Recipes

Joey's Shakshuka (serves 6-8)

- E. Eggs (2 per person)
- On. Onion (1 big)
- TP. Tomato Paste (4-6 oz)
- TC. Canned tomatoes (56 oz)
- Oo. Olive oil
- F. Feta cheese
- G. Cookable green (spinach, swiss chard, etc.)
- V. Eggplant and/or other veggie
- C. Cumin
- Gr. Parsley/Cilantro/Lemon
- SP Fresh serrano pepper
- P. Pita



1. Tomato sauce: if TC are whole, mash them. Add TP. Put in "Crest" - casserole pan.
2. Sauté onion (On) and Veggies (V) in olive oil^(Oo). When almost cooked, add Cumin (C). Add to crest. Simmer for ≈ 40 mins.
3. Prepare garnishes: cut parsley, cilantro, lemon (Gr), cut serrano pepper (SP), and serve along with Pita (P) and Feta (F).
4. About minutes before eating, add eggs (E) uncooked to crest, where they'll poach. A few minutes later, add greens (G). Serve when cooked.

Summary on operads, and a quick word on algebras

Operads show up everywhere.

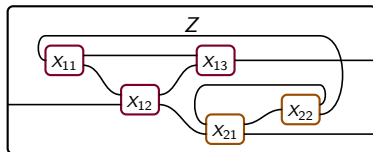
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Summary on operads, and a quick word on algebras

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I didn't talk about algebras, but they're important too.



The operad is syntax, the algebra is semantics: something it can be about.

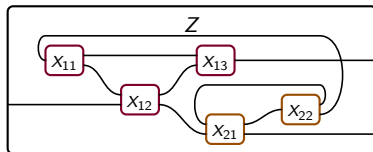
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So far, all of our arrangements have been static. Let's relax that.

Outline

- 1 Introduction
- 2 Operads: a framework for compositional operations
- 3 **Dynamic operads**
 - Dynamically changing arrangements
 - Dynamic operads
- 4 Conclusion

Even machines change over time

People often refer to functions as machines.

- A function $f: A \rightarrow B$ takes in A 's and spits out B 's.
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- An over-used key on your keyboard might have a faded letter.
- Your shoes wear down according to how you walk.
- Similarly for your baseball glove, your brain, your home.
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I want to call such a thing a *dynamic function* $A \rightarrow B$.

- They're modeled by *Mealy machines*, i.e...
- ...a set S of "states" and a function $f: S \times A \rightarrow B \times S$.
- For any state $s: S$, you get a machine that takes an $a: A$ and...
- ...not only gives out a B but also updates the state.
- The keys, shoes, glove, brain, and home respond to the input...
- ...and can also be changed by it.

Deep learning, prediction markets, organizations

We'll soon see how to similarly generalize operads. But first, applications.

- In training artificial neural networks:
 - The state is the current weights and biases of the ANN.
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We (joint with B.T. Shapiro) have a math model of the first two.

- Also (joint with S. Libkind) a simplified model of Hebbian learning.
- These are all examples of a single mathematical structure.

Dynamic arrangements

Recall that operads were systems for **nestable arrangements**, e.g.:

- Wiring diagrams (WDs), protein materials, probability distributions, ...
- ... grammars, recipes, etc.

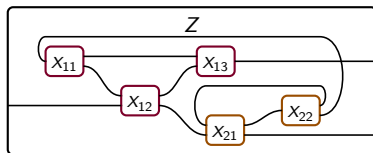
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In each case, what would it mean for the **arrangements** to be dynamic?

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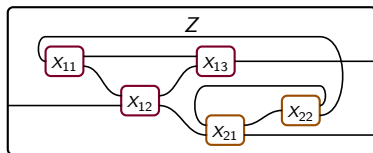
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The state controls the **arrangement**, and it changes based on what occurs.

Dynamic operads

A *dynamic operad*² is a coherent system of dynamic **arrangements**.

- The coherence means: the changing **arrangements** **nest** lawfully.
- Examples: deep learning, Hebbian learning, prediction markets,...

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- Deep learning: basically the chain rule.
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I would love a dynamic operad of wiring diagrams, but I can't think of one.

- How we arrange themselves—who you hang out with—is v. important.
- And same at all levels: certain things like to work together.
- Shouldn't there be some coherent system for this?

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Hopefully this gives a taste of ACT. It's really fun and increasingly useful.

Thanks! Comments and questions welcome...