The double category $\mathbb{I}\mathbf{nt}(\mathbf{Poly}_+)$ models control flow and data flow

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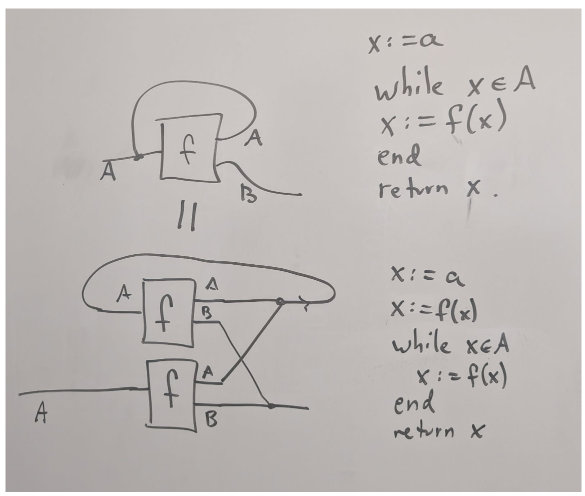
Joint work in progress, with Grigory Kondyrev @ Noeon Research

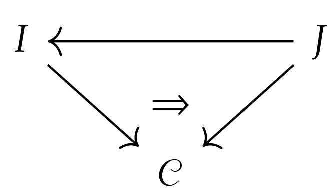
I. Introduction

A. Control flow as while loops (cocartesian traces) 3 mins



B. Data flow as "query" morphisms 3 mins

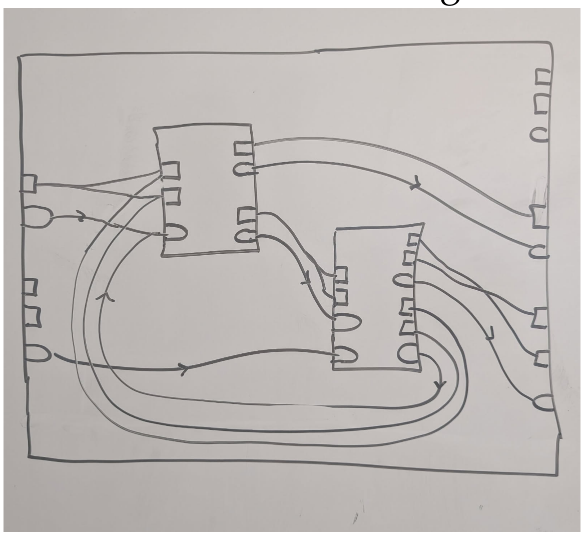
1. Morphisms between formal limits within an ontology
2. Roughly Diag($\mathcal{C}$),



3. e.g., when $\mathcal{C} = 3$, we have the query morphism

$$X_1 \times X_2 \times X_3 \to X_1 \times X_1 \times X_3$$

C. Control and data flow diagrams 3 mins



D. Plan for the talk 1 min

1. Uniform traced monoidal categories $\mathcal{U}$
2. The thin cocartesian double category $\mathbb{I}\mathbf{nt}(\mathcal{U})$
3. $\mathbb{I}\mathbf{nt}(\mathbf{Poly}_\star)$ and $\mathbb{I}\mathbf{nt}(c\text{-}\mathbf{Set}[d]_\star)$.

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II. Uniform traced monoidal categories $\mathcal{U}$

A. Traced symmetric monoidal categories 5 mins

1. String diagrams for monoidal categories
2. String diagrams for trace
3. Example: $(\mathbf{Set}_\star, 0_\star, +_\star)$, $(\mathbf{FDVect}_\mathbb{R}, \mathbb{R}, \otimes)$.
4. Any full (monoidal) subcat'y of a traced cat'y is traced.

B. Uniformity (Hasegawa 2004 The Uniformity Principle on Traced Monoidal Categories) 4 mins

$$\begin{array}{ccc} A+U & \xrightarrow{f} & B+U \\ {\scriptstyle A+h}\downarrow & \checkmark & \downarrow{\scriptstyle B+h} \\ A+V & \xrightarrow[g]{} & B+V \end{array} \quad \Rightarrow \quad \mathrm{Tr}^U_{A,B}(f) = \mathrm{Tr}^V_{A,B}(g) : A \to B$$

C. The category $\mathbf{Set}_\star$ is uniform. 4 mins

D. If $\mathcal{U}$ is uniform traced, so is $\mathbf{Fun}(\mathcal{C}, \mathcal{U})$ for any $\mathcal{C}$. 8 mins

1. Monoidal structure is pointwise $(X \otimes Y)(c) \coloneqq X(c) \otimes Y(c)$.
2. Trace is also pointwise: given $f\colon U \otimes X \Rightarrow U \otimes Y$,

$$\mathrm{Tr}^{U}_{X,Y}(f)_c \coloneqq \mathrm{Tr}^{U(c)}_{X(c),Y(c)}(f_c).$$

3. So $\mathbf{Fun}(-, \mathcal{U})\colon \mathbf{Cat}^{\mathrm{op}} \to \text{TracedMonCat}$.
4. Any full subcategory of a uniform traced cat'y is uniform traced.

E. More examples of uniform traced. 5 mins

1. $(\mathbf{Poly}_\star, 0_\star, +_\star)$ is UT: a full subcategory of $\mathbf{Fun}(\mathbf{Set}, \mathbf{Set}_\star)$.
2. $\mathbf{Set}[\mathcal{D}]_\star \subseteq \mathbf{Fun}(\mathcal{D}\text{-}\mathbf{Set}, \mathbf{Set}_\star)$ full subcat'y
3. Example: $\mathbf{Set}[y_1, \ldots, y_L]_\star$, pointed polynomials in L-variables, e.g. $x^2 + \mathbb{N}xy^{\mathbb{R}}z^2 + 1$.
4. $\mathcal{C}\text{-}\mathbf{Set}[\mathcal{D}]_\star = \mathbf{Fun}(\mathcal{C}, \mathbf{Set}[\mathcal{D}]_\star)$.

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III. The thin cocartesian double category $\mathbb{I}\mathbf{nt}(\mathcal{U})$

A. The **Int** constr'n: $\mathbf{Int}(\mathcal{T})$ is free CC cat'y on $\mathcal{T}$. 6 mins

1. Objects: $\{(X^i, X^o) \in \mathrm{Ob}(\mathcal{T})^2\}$
2. Morphisms $\varphi\colon (X^i, X^o) \nrightarrow (Y^i, Y^o)$ in $\mathbf{Int}(\mathcal{T})$ means

$$Y^i + X^o \to Y^o + X^i$$

   Idea: X is *inside* Y. Example, $X = (0, 0)$ means nothing inside.
3. Composition. Given also $Z^i + Y^o \to Z^o + Y^i$, get $\varphi \mathbin{⨟} \psi$ by

$$\mathrm{Tr}^{Y^i}_{Z^i+X^o, Z^o+X_i}\left(Z^i+Y^i+X^o \to Z^i+Y^o+X^i \to Z^o+Y^i+X^i\right)$$

B. The double category $\mathbb{I}\mathbf{nt}(\mathcal{U})$ for $\mathcal{U}$ uniform 5 mins

1. Definition of $\mathbb{I}\mathbf{nt}(\mathcal{U})$.
   a. Tight direction is $\mathcal{U} \times \mathcal{U}$.
   b. Loose arrows are $\mathbf{Int}(\mathcal{U})$ as above.
   c. Cells are commuting diagrams

$$\begin{array}{ccc} Y_1^i + X_1^o & \xrightarrow{\varphi} & Y_1^o + X_1^i \\ \downarrow & & \downarrow \\ Y_2^i + X_2^o & \xrightarrow[\psi]{} & Y_2^o + X_2^i \end{array}$$

2. Thin. They compose horizontally by uniformity.

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IV. $\mathbb{I}\mathbf{nt}(\mathbf{Poly}_\star)$ and $\mathbb{I}\mathbf{nt}(\mathcal{C}\text{-}\mathbf{Set}[\mathcal{D}]_\star)$.

A. Consider $\mathbb{I}\mathbf{nt}(\mathbf{Poly}_\star)$ and $\mathbb{I}\mathbf{nt}(\mathcal{C}\text{-}\mathbf{Set}[\mathcal{D}]_\star)$ 5 mins

1. Both are cocartesian double categories
2. Compact closed in the horizontal direction.
3. Control and data flow diagrams

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Bonus material

B. A factorization system and a weird property 10 mins

1. Vertical maps have an OFS:

$$(X^i, X^o) \xrightarrow{L} (X^i, Y^o) \xrightarrow{R} (Y^i, Y^o).$$

2. When $\mathcal{U}$ has pushouts and an OFS with extensive right class...
   a. For example, $\mathbf{Set}_\star$ has “restriction category” OFS

b. $X \rightharpoonup Y$ factors as “domain map” $X \rightharpoonup X'$ followed by total function $X' \to Y$

3. ...we get the following property. Any diagram

$$
\begin{array}{ccc}
(X_1^i, X_1^o) & \relbar\joinrel\mapstochar\joinrel\rightarrow & (Y_1^i, Y_1^o) \\
{\scriptstyle L}\big\downarrow & & \big\downarrow{\scriptstyle R} \\
(X_1^i, X_2^o) & \Downarrow & (Y_2^i, Y_1^o) \\
\big\downarrow & & \big\downarrow \\
(X^i, X^o) & \relbar\joinrel\mapstochar\joinrel\rightarrow & (Y^i, Y^o)
\end{array}
$$

can be factored as

$$
\begin{array}{ccc}
(X_1^i, X_1^o) & \relbar\joinrel\mapstochar\joinrel\rightarrow & (Y_1^i, Y_1^o) \\
{\scriptstyle L}\big\downarrow & & \big\downarrow{\scriptstyle R} \\
(X_1^i, X_2^o) & \Downarrow & (Y_2^i, Y_1^o) \\
{\scriptstyle R}\big\downarrow & & \big\downarrow{\scriptstyle L} \\
(X_2^i, X_2^o) & \relbar\joinrel\mapstochar\joinrel\rightarrow & (Y_2^i, Y_2^o) \\
\big\downarrow & \Downarrow & \big\downarrow \\
(X^i, X^o) & \relbar\joinrel\mapstochar\joinrel\rightarrow & (Y^i, Y^o)
\end{array}
$$

4. Indeed, the top diagram boils down to

$$
\begin{array}{ccc}
Y_1^i + X_1^o & \longrightarrow & Y_1^o + X_1^i \\
\big\downarrow & & \big| \\
Y_2^i + X_2^o & & \big| \\
\big\downarrow & & \big\downarrow \\
Y^i + X^o & \longrightarrow & Y^o + X^i
\end{array}
$$

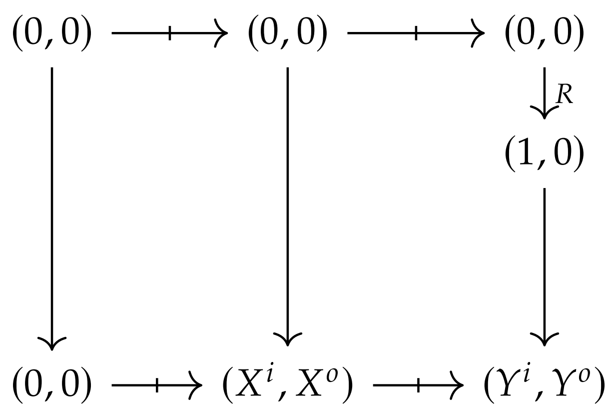
To find the universal thing:

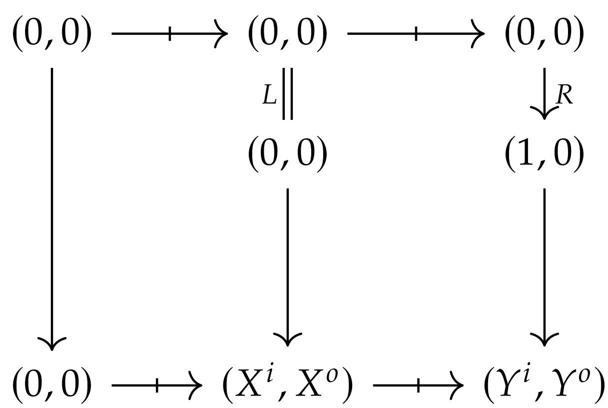
a. Perform a pushout

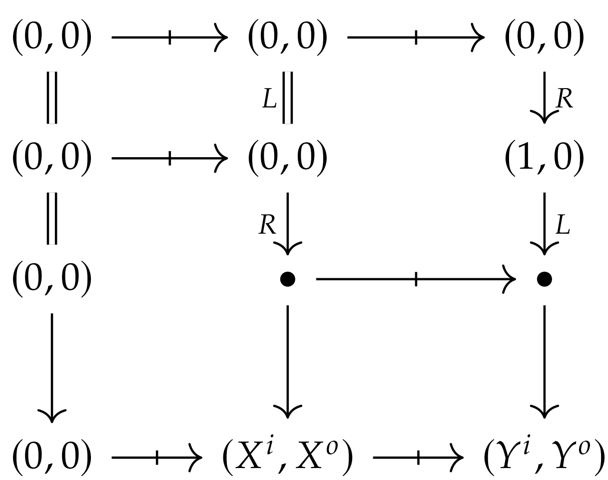
b. Factor the right map

c. Use extensivity to form new pair of objects.

C. Use the above to get trajectories 4 mins







$$
\begin{tikzcd}
(0,0) \arrow[r, "\shortmid" marking] \arrow[dd] & (0,0) \arrow[r, "\shortmid" marking] \arrow[d, equal, "L"'] & (0,0) \arrow[d, "R"] \\
 & (0,0) \arrow[d] & (1,0) \arrow[d] \\
(0,0) \arrow[r, "\shortmid" marking] & (X^i, X^o) \arrow[r, "\shortmid" marking] & (Y^i, Y^o)
\end{tikzcd}
$$

$$
\begin{tikzcd}
(0,0) \arrow[r, "\shortmid" marking] \arrow[d, equal] & (0,0) \arrow[r, "\shortmid" marking] \arrow[d, equal, "L"'] & (0,0) \arrow[d, "R"] \\
(0,0) \arrow[r, "\shortmid" marking] \arrow[d, equal] & (0,0) \arrow[d, "R"'] & (1,0) \arrow[d, "L"] \\
(0,0) \arrow[d, equal] & \bullet \arrow[r, "\shortmid" marking] \arrow[d, "L"'] & \bullet \arrow[dd] \\
(0,0) \arrow[r, "\shortmid" marking] \arrow[d] & \bullet \arrow[d] & \\
(0,0) \arrow[r, "\shortmid" marking] & (X^i, X^o) \arrow[r, "\shortmid" marking] & (Y^i, Y^o)
\end{tikzcd}
$$

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