Categories by Kan extension

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Outline

Categories by Kan extension?

The left Kan extension of p along q is a *functor*. How can it be a *category*?

- Answer: It can be a comonad, e.g. a *density comonad*, and
- ... certain comonads c can be id'd with cat'ies $\mathcal{C}$,[1] with $\mathbf{Set}^c \cong \mathbf{Set}^{\mathcal{C}}$.
- I'll explain both parts of this story in the talk.

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Application: nerve constructions and Lawvere theories (jww B. Shapiro).

- Example: let T be the free category monad on graphs.
- Then $\mathrm{Lan}_{T \circ T} T$ "is" Δ^{op}, whose copresheaves are simplicial sets.
- For any T-algebra $\mathcal{C}$, i.e. category, its nerve is induced by the UP.

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Plan of the talk

The plan of the talk is as follows:

- Recall basic theory of left Kan extensions and intro. $\begin{bmatrix} q \\ p \end{bmatrix}$ notation.
- Explain the "comonads as categories" idea.
- Give the "categories by Kan extension" construction.
- Discuss several examples, e.g. nerves (jww Brandon Shapiro)

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To keep statements simple, we'll elide mention of accessibility[2] in the talk.

[2] An object $c \in \mathcal{C}$ is κ-*small* if $\mathcal{C}(c, -)$ preserves κ-filtered colimits. A category is *accessible* if it has κ-filtered colimits and is generated under them by κ-small objects, for some regular cardinal κ. A functor is acc'ble if it preserves κ-filtered colimits for some κ.

Outline

Left Kan extensions

Given functors $q\colon \mathcal{C} \to \mathcal{C}'$ and $p\colon \mathcal{C} \to \mathcal{D}$, with $\mathcal{D}$ cocomplete, write

$$\begin{tikzcd} \mathcal{C} \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, phantom, "\Downarrow"] & \mathcal{D} \\ \mathcal{C}' \arrow[ur, bend right, "{\begin{bmatrix} q \\ p \end{bmatrix}}"'] & \end{tikzcd} \qquad \begin{bmatrix} q \\ p \end{bmatrix} := \mathrm{Lan}_q(p).$$

It is the left adjoint to precomposition by q. For $r\colon \mathcal{C}' \to \mathcal{D}$,

$$\mathrm{Hom}\left(\begin{bmatrix} q \\ p \end{bmatrix}, r\right) \cong \mathrm{Hom}(p, r \circ q).$$

It comes with unit $p \Rightarrow \begin{bmatrix} q \\ p \end{bmatrix} \circ q$ and counit $\begin{bmatrix} q \\ r \circ q \end{bmatrix} \Rightarrow r$.

Left Kan calculus

Intuitively, $\begin{bmatrix} q \\ p \end{bmatrix}$ acts like inner hom. Here are some examples:

- Variance: $P \to P'$ and $Q \leftarrow Q'$ induce $[Q, P] \to [Q', P']$

$$\begin{array}{ccc} q & \longleftarrow & q' \\ p & \longrightarrow & p' \end{array} \quad \rightsquigarrow \quad \begin{bmatrix} q \\ p \end{bmatrix} \to \begin{bmatrix} q' \\ p' \end{bmatrix}$$

- Unit and tensor: $P \cong [I, P]$ and $[Q', [Q, P]] \cong [Q' \otimes Q, P]$.

$$p \cong \begin{bmatrix} \mathrm{id}_{\mathcal{C}} \\ p \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} q' \\ \begin{bmatrix} q \\ p \end{bmatrix} \end{bmatrix} \cong \begin{bmatrix} q' \circ q \\ p \end{bmatrix}.$$

- Duals: if $\ell \dashv r$ then: e.g. duals in a CCC: $[P \otimes V, Q] \cong [P, Q \otimes V^*]$.

$$\begin{bmatrix} q \\ p \circ r \end{bmatrix} \cong \begin{bmatrix} q \circ \ell \\ p \end{bmatrix}.$$

- Cocomposition: $I \to [p, p]$ and $[p, q] \otimes [q, r] \to [p, r]$

$$\begin{bmatrix} p \\ p \end{bmatrix} \to \mathrm{id}_{\mathcal{D}} \qquad \text{and} \qquad \begin{bmatrix} r \\ p \end{bmatrix} \to \begin{bmatrix} q \\ p \end{bmatrix} \circ \begin{bmatrix} r \\ q \end{bmatrix}$$

Outline

Pra-functors between copresheaf categories

A *parametric right adjoint (pra)functor* $p\colon \mathbf{Set}^{\mathcal{C}} \to \mathbf{Set}^{\mathcal{D}}$ is

- one such that $p/1\colon \mathbf{Set}^{\mathcal{C}}/1 \to \mathbf{Set}^{\mathcal{D}}/p(1)$ is a right adjoint, or *equivalently*
- a connected-limit preserving functor, or *equivalently*
- a functor from $\mathcal{D}$ to the coproduct completion of $(\mathbf{Set}^{\mathcal{C}})^{\mathsf{op}}$
  - Given $(I, X)\colon \mathcal{D} \to \mathbf{Fam}\big((\mathbf{Set}^{\mathcal{C}})^{\mathsf{op}}\big)$, use $p(X')(d) := \sum_{i \in I_d} \mathbf{Set}^{\mathcal{C}}(X_i, X')$

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Suppose $p\colon \mathbf{Set}^{\mathcal{C}} \to \mathbf{Set}^{\mathcal{D}}$ is a prafunctor, i.e. $\mathcal{D} \xrightarrow{p} \mathbf{Fam}\big((\mathbf{Set}^{\mathcal{C}})^{\mathsf{op}}\big)$

- For *any* $\mathbf{Set}^{\mathcal{C}} \xrightarrow{q} \mathbf{Set}^{\mathcal{C}'}$, the left Kan extension $\begin{bmatrix} q \\ p \end{bmatrix}$ is a prafunctor:

$$\begin{tikzcd}
\mathbf{Set}^{\mathcal{C}} \arrow[r, "\overset{\text{pra}}{p}"] \arrow[d, "\text{any } q"'] \arrow[dr, phantom, "\Downarrow"] & \mathbf{Set}^{\mathcal{D}} \\
\mathbf{Set}^{\mathcal{C}'} \arrow[ur, bend right, "\begin{bmatrix} q \\ p \end{bmatrix} \text{ pra}"'] &
\end{tikzcd}$$

- Indeed, compose $\mathcal{D} \to \mathbf{Fam}\big((\mathbf{Set}^{\mathcal{C}})^{\mathsf{op}}\big) \to \mathbf{Fam}\big((\mathbf{Set}^{\mathcal{C}'})^{\mathsf{op}}\big)$.

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A few more facts:

- Any right adjoint $\mathbf{Set}^{\mathcal{C}} \to \mathbf{Set}^{\mathcal{D}}$ is pra.
- elts: $\mathbf{Set}^{\mathcal{C}} \to \mathbf{Set}$ is pra, and has a right adjoint $\mathbf{Set} \to \mathbf{Set}^{\mathcal{C}}$, namely
- $X \mapsto C \mapsto \mathbf{Set}(H_C, X)$, where $H_C = \sum_{C' \in \mathrm{Ob}\,\mathcal{C}} \mathcal{C}(C, C') = \mathrm{elts}(よ^{C})$, "maps out"

Pra-comonads are categories

For any categories $\mathcal{C}, \mathcal{D}$, let $\mathbb{P}(\mathcal{C}, \mathcal{D}) \coloneqq \mathbf{Pra}(\mathbf{Set}^{\mathcal{D}}, \mathbf{Set}^{\mathcal{C}})$.

- **Poly** $\coloneqq \mathbb{P}(1, 1)$. (And Gambino-Kock's $\mathbb{P}\mathbf{olyFun}_{\mathbf{Set}} = \mathbb{P}\big|_{discCat}$.)
- Limit compl'n LC distributes over **Fam**, and Kleisli is $\mathbb{P} \cong \mathbb{C}\mathbf{at}_{\mathbf{Fam} \circ \mathsf{LC}}$.

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Categories = Polynomial comonads = Comonads in $\mathbb{P}(1, 1)$.

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- You can read off the objects and morphisms from the poly'l form.

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More generally, a comonad $\mathcal{D} \xrightarrow{c} \mathcal{D}$ in $\mathbb{P}$ has an underlying category.

- Conjugate $\mathbf{Set} \xrightarrow{H} \mathbf{Set}^{\mathcal{D}} \xrightarrow{c} \mathbf{Set}^{\mathcal{D}} \xrightarrow{\text{elts}} \mathbf{Set}$ is a polynomial comonad.
- Hence we get a category from any comonad c in $\mathbb{P}$.
- The counit $c \to \mathrm{id}_{\mathcal{D}}$ gives a poly comonad map $\text{elts} \circ c \circ H \to d$
- Maps of poly'l comonads correspond to *cofunctors* between cat'ies.

Pra-comonads are categories

For any categories $\mathcal{C}, \mathcal{D}$, let $\mathbb{P}(\mathcal{C}, \mathcal{D}) := \mathbf{Pra}(\mathbf{Set}^{\mathcal{D}}, \mathbf{Set}^{\mathcal{C}})$.

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Categories $=$ Polynomial comonads $=$ Comonads in $\mathbb{P}(1, 1)$.

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So let's find some comonads in $\mathbb{P}$.

Outline

Density comonads

For any $p\colon \mathbf{Set}^{\mathscr{C}} \to \mathbf{Set}^{\mathscr{D}}$, the left Kan extension $\begin{bmatrix} p \\ p \end{bmatrix}$ is a comonad.

- For closures, $[p, p]$ is a monoid by eval: $[p, p] \otimes [p, p] \to [p, p]$.
- Similarly, using coeval $\begin{bmatrix} r \\ p \end{bmatrix} \to \begin{bmatrix} q \\ p \end{bmatrix} \circ \begin{bmatrix} r \\ q \end{bmatrix}$ gives $\begin{bmatrix} p \\ p \end{bmatrix} \to \begin{bmatrix} p \\ p \end{bmatrix} \circ \begin{bmatrix} p \\ p \end{bmatrix}$.
- $\begin{bmatrix} p \\ p \end{bmatrix}\colon \mathbf{Set}^{\mathscr{D}} \to \mathbf{Set}^{\mathscr{D}}$ is called the *density* comonad of p.

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For p pra, we said $\begin{bmatrix} p \\ p \end{bmatrix}$ can be id'd with a cat'y and a cofunctor $\begin{bmatrix} p \\ p \end{bmatrix} \to \mathcal{D}$.

- That cofunctor is the identity iff p is dense, hence the name.
- We won't care about that case. We want to construct new categories.

Examples:

- Let $\mathit{list}\colon \mathbf{Set} \to \mathbf{Set}$ be $\sum_{N:\mathbb{N}} y^N$. Then $\begin{bmatrix} \mathit{list} \\ \mathit{list} \end{bmatrix} \cong \mathbf{FinSet}^{\mathrm{op}}$.
- The maybe monad $y + 1$ has $\begin{bmatrix} y+1 \\ y+1 \end{bmatrix} \cong \boxed{\bullet \longrightarrow \bullet}$, the walking arrow.

Finding density comonads

Theorem

Suppose given a diagram of the following form, where k, p are pra:

$$\begin{tikzcd}
\mathbf{Set}^{\mathcal{C}} \arrow[r, "k"] \arrow[d, "t"'] & \mathbf{Set}^{\mathcal{C}} \arrow[r, "p"] \arrow[d, phantom, "\Downarrow"] & \mathbf{Set}^{\mathcal{D}} \\
\mathbf{Set}^{\mathcal{C}} \arrow[d, "p"'] & {} & \\
\mathbf{Set}^{\mathcal{D}} \arrow[rruu, bend right, "{\begin{bmatrix} p\circ t \\ p\circ k \end{bmatrix}}"'] & &
\end{tikzcd}$$

If k is a comonad and t is a monad and there is a distributive law $t\circ k \to k\circ t$ then the Kan extension $\begin{bmatrix} p\circ t \\ p\circ k \end{bmatrix}$ is a comonad.

E.g., if $k = \mathrm{id}$ or $t = \mathrm{id}$ then unique dist. law, so $\begin{bmatrix} p\circ t \\ p \end{bmatrix}$ & $\begin{bmatrix} p \\ p\circ k \end{bmatrix}$ comonads.

- Why is $\begin{bmatrix} p\circ t \\ p\circ k \end{bmatrix}$ a comonad? Factor the non-id as $\mathbf{Set}^{\mathcal{C}} \leftrightarrows \mathbf{EM} \to \mathbf{Set}^{\mathcal{C}} \ldots$
- … and use $\begin{bmatrix} p \\ p\circ\ell\circ r \end{bmatrix} = \begin{bmatrix} p\circ\ell \\ p\circ\ell \end{bmatrix}$ or $\begin{bmatrix} p\circ r\circ\ell \\ p \end{bmatrix} = \begin{bmatrix} p\circ r \\ p\circ r \end{bmatrix}$, respectively.

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Lawvere theories

Much of the rest is joint work with Brandon Shapiro.[3]

Let t be a finitary monad on $\mathbf{Set}$. It has a Lawvere theory $\mathbf{Law}_t$.

- Product-preserving functors $\mathbf{Law}_t \to \mathbf{Set}$ are t-algebras.
- Mouthful: $\mathbf{Law}_t$ is the opposite of the full subcategory of the Kleisli category $\mathbf{Set}_t$ spanned by the finite sets.

[3] *A Polynomial Construction of Nerves for Higher Categories*. 2405.13157

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- Easier: $\mathbf{Law}_t \cong \begin{bmatrix} \mathit{list} \circ t \\ \mathit{list} \end{bmatrix}$. Shown above: this is poly'l, even if t is not.
- Example: $\mathbf{Law}_{\mathrm{id}} \cong \mathbf{FinSet}^{\mathrm{op}} \cong \begin{bmatrix} \mathit{list} \\ \mathit{list} \end{bmatrix}$.
- If t instead is κ-ary & S-sorted, let $\ell := \sum_{|A|<\kappa} y^A$. $\mathbf{Law}_t = \begin{bmatrix} \ell \circ Sy \circ t \\ \ell \circ Sy \end{bmatrix}$.

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Given a t-algebra $\alpha\colon t \circ X \to X$, what's the connection?

$$
\begin{tikzcd}
1 \arrow[r, "X"] \arrow[drr, bend right, "X"'] & \mathbf{Set} \arrow[rrr, "list"] \arrow[dr, "t"] \arrow[d, phantom, "\Downarrow\alpha"] & & & \mathbf{Set} \\
& {} & \mathbf{Set} \arrow[r, "list"'] & \mathbf{Set} \arrow[ur, "{\begin{bmatrix} list\circ t \\ list \end{bmatrix}}"'] \arrow[ul, phantom, "\Downarrow"] &
\end{tikzcd}
$$

$list \circ X$ is a $\begin{bmatrix} list\circ t \\ list \end{bmatrix}$-coalgebra i.e. a $\mathbf{Law}_t$-copresheaf with carrier $list(X)$.

- i.e. a copresheaf $(N \mapsto X^N)\colon \mathbf{Law}_t \to \mathbf{Set}$. Its elements: $list(X)$.

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Arbitrary product completions

The product completion of a category is dual$^?$ to Lawvere construction.

- Let $\mathcal{C}$ be a category and c the corresponding polynomial comonad.
- Then $\begin{bmatrix} \mathit{list} \\ \mathit{list} \circ c \end{bmatrix}$ is the finite product completion of c.
- It's kinda dual to Lawvere theory: $\begin{bmatrix} \mathit{list} \circ t \\ \mathit{list} \end{bmatrix}$ vs. $\begin{bmatrix} \mathit{list} \\ \mathit{list} \circ c \end{bmatrix}$.

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You can ask for whatever "products" you want.

- $\begin{bmatrix} y^3+y^7 \\ (y^3+y^7) \circ c \end{bmatrix}$ is cat'y of 3-ary and 7-ary products in $\mathcal{C}$.
- This sounds weird, but the construction is at least well-behaved.

There's a normal lax double functor $\begin{bmatrix} p \\ p \circ c \end{bmatrix} : \mathbb{C}\mathbf{odisc}(\mathbf{Poly}^{\mathbf{cart}}) \times \mathbb{P}\mathbf{rof} \to \mathbb{P}\mathbf{rof}$.

- Here $\mathbb{C}\mathbf{odisc}(\mathbf{Poly}^{\mathbf{cart}})$ has tight $:= \mathbf{Poly}^{\mathbf{cart}}$, loose $:=$ codiscrete.

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- For each p, q and profunctor $\mathcal{C} \nrightarrow \mathcal{D}$, get profunctor $\begin{bmatrix} p \\ p \circ c \end{bmatrix} \nrightarrow \begin{bmatrix} q \\ q \circ d \end{bmatrix}$.

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- For each p, q and profunctor $\mathcal{C} \nrightarrow \mathcal{D}$, get profunctor $\begin{bmatrix} p \\ p \circ c \end{bmatrix} \nrightarrow \begin{bmatrix} q \\ q \circ d \end{bmatrix}$.
- In fact, the action is lax $\circ$-monoidal: $\begin{bmatrix} p \\ p \circ \begin{bmatrix} q \\ q \circ - \end{bmatrix} \end{bmatrix} \longrightarrow \begin{bmatrix} p \circ q \\ p \circ q \circ - \end{bmatrix}$.
- So since *list* is a cartesian monad, so is product completion $\begin{bmatrix} \mathit{list} \\ \mathit{list} \circ - \end{bmatrix}$.

Nerves

The nerve of a category is a copresheaf on Δ^{op}. Let's find Δ^{op}.

- A category is a graph for which each path of edges is assigned an edge
- That is $\mathcal{C}$ is an algebra for the *paths* monad on **Grph** $\cong$ $\mathbf{Set}^{\bullet \rightrightarrows \bullet}$.

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- That is $\mathcal{C}$ is an algebra for the *paths* monad on $\mathbf{Grph} \cong \mathbf{Set}^{\bullet \rightrightarrows \bullet}$.
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\begin{tikzcd}
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Nerves

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\end{tikzcd}$$

- This makes $\mathit{paths}(\mathcal{C})$ a coalgebra, i.e. a copresheaf, on $\begin{bmatrix} \mathit{paths} \circ \mathit{paths} \\ \mathit{paths} \end{bmatrix}$.
- And indeed, $\begin{bmatrix} \mathit{paths} \circ \mathit{paths} \\ \mathit{paths} \end{bmatrix} \cong \Delta^{\mathrm{op}}$. Its copresheaves are simplicial sets.
- Also $\begin{bmatrix} \mathit{paths} \\ \mathit{paths} \end{bmatrix} \cong \Delta^{\mathrm{op}}_{\mathrm{inert}}$, and the unit induces $\begin{bmatrix} \mathit{paths} \\ \mathit{paths} \end{bmatrix} \xrightarrow{\mathrm{io}} \begin{bmatrix} \mathit{paths} \circ \mathit{paths} \\ \mathit{paths} \end{bmatrix}$

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Same idea works for higher cat'l structures like nerves of double cat'ies.

- Pick *cell-shape* category $\mathcal{G}$, *compositions* pra-monad $t : \mathbf{Set}^{\mathcal{G}} \to \mathbf{Set}^{\mathcal{G}}$
- Nerves of that sort of thing are $\begin{bmatrix} t\circ t \\ t \end{bmatrix}$-coalgebras, i.e. copresheaves.

Outline

Summary

Thinking about categories as comonads opens up a new source of cat'ies.

- As density comonads: certain Kan extensions give rise to categories.
- If p, k pra, $t \circ k \to k \circ t$ monad-comonad distr, then $\begin{bmatrix} p \circ t \\ p \circ k \end{bmatrix}$ comonad.
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Lawvere theories, nerve theories, and finite product compl'ns arise this way.

- If you want to see, e.g. monoidal double cat'ies, as presheaves on sth.
- Write out your cell shape cat'y $\mathcal{G}$ and compositions as $t : \mathbb{P}(\mathcal{G}, \mathcal{G})$.
- Then the comonad $\Theta_t^{\mathrm{op}} := \begin{bmatrix} t \circ t \\ t \end{bmatrix}$ is a category.
- Every algebra (monoidal double cat'y) is a Θ_t-presheaf.
- The Lawvere theory $\begin{bmatrix} \mathit{list} \circ t \\ \mathit{list} \end{bmatrix}$ is always a category, since *list* is pra.

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Quick plug: "Comonads on **Set** are spaces!" See A. Fairbanks's talk.

Thank you for your time; questions and comments welcome.

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