

Categories by Kan extension

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Outline

1 Introduction

- Categories by Kan extension?
- Plan of the talk

2 Basic theory of left Kan extensions

3 Comonads as categories

4 Categories by Kan extension

5 Examples

6 Conclusion

Categories by Kan extension?

The left Kan extension of p along q is a *functor*. How can it be a *category*?

- Answer: It can be a comonad, e.g. a *density comonad*, and
- ... certain comonads c can be id'd with cat'ies $\mathcal{C}$,¹ with $\mathbf{Set}^c \cong \mathbf{Set}^{\mathcal{C}}$.
- I'll explain both parts of this story in the talk.

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Application: nerve constructions and Lawvere theories (jww B. Shapiro).

- Example: let T be the free category monad on graphs.
- Then $\mathrm{Lan}_{T \circ T} T$ “is” Δ^{op} , whose copresheaves are simplicial sets.
- For any T -algebra $\mathcal{C}$, i.e. category, its nerve is induced by the UP.

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Plan of the talk

The plan of the talk is as follows:

- Recall basic theory of left Kan extensions and intro. $\begin{bmatrix} q \\ p \end{bmatrix}$ notation.
- Explain the “comonads as categories” idea.
- Give the “categories by Kan extension” construction.
- Discuss several examples, e.g. nerves (jww Brandon Shapiro)

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To keep statements simple, we'll elide mention of accessibility² in the talk.

²An object $c \in \mathcal{C}$ is κ -small if $\mathcal{C}(c, -)$ preserves κ -filtered colimits. A category is *accessible* if it has κ -filtered colimits and is generated under them by κ -small objects, for some regular cardinal κ . A functor is acc'ble if it preserves κ -filtered colimits for some κ .

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- 1 Introduction
- 2 Basic theory of left Kan extensions**
 - Left Kan extensions
 - Left Kan calculus
- 3 Comonads as categories
- 4 Categories by Kan extension
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Left Kan extensions

Given functors $q: \mathcal{C} \rightarrow \mathcal{C}'$ and $p: \mathcal{C} \rightarrow \mathcal{D}$, with $\mathcal{D}$ cocomplete, write

$$\begin{array}{ccc}
 \mathcal{C} & \xrightarrow{p} & \mathcal{D} \\
 q \downarrow & \Downarrow & \uparrow [q] \\
 \mathcal{C}' & \xrightarrow{\quad} & \mathcal{D}
 \end{array}
 \quad [q]_p := \text{Lan}_q(p).$$

It is the left adjoint to precomposition by q . For $r: \mathcal{C}' \rightarrow \mathcal{D}$,

$$\text{Hom}\left(\left[\begin{smallmatrix} q \\ p \end{smallmatrix}\right], r\right) \cong \text{Hom}(p, r \circ q).$$

It comes with unit $p \Rightarrow \left[\begin{smallmatrix} q \\ p \end{smallmatrix}\right] \circ q$ and counit $\left[\begin{smallmatrix} q \\ r \circ q \end{smallmatrix}\right] \Rightarrow r$.

Left Kan calculus

Intuitively, $\begin{bmatrix} q \\ p \end{bmatrix}$ acts like inner hom. Here are some examples:

- **Variance:** $P \rightarrow P'$ and $Q \leftarrow Q'$ induce $[Q, P] \rightarrow [Q', P']$

$$\begin{array}{ccc} q & \longleftarrow & q' \\ p & \longrightarrow & p' \end{array} \quad \rightsquigarrow \quad \begin{bmatrix} q \\ p \end{bmatrix} \rightarrow \begin{bmatrix} q' \\ p' \end{bmatrix}$$

- **Unit and tensor:** $P \cong [I, P]$ and $[Q', [Q, P]] \cong [Q' \otimes Q, P]$.

$$p \cong \begin{bmatrix} \text{id}_P \\ p \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} q' \\ \begin{bmatrix} q \\ p \end{bmatrix} \end{bmatrix} \cong \begin{bmatrix} q' \circ q \\ p \end{bmatrix}.$$

- **Duals:** if $\ell \dashv r$ then: e.g. duals in a CCC: $[P \otimes V, Q] \cong [P, Q \otimes V^*]$.

$$\begin{bmatrix} q \\ p \circ r \end{bmatrix} \cong \begin{bmatrix} q \circ \ell \\ p \end{bmatrix}.$$

- **Cocomposition:** $I \rightarrow [p, p]$ and $[p, q] \otimes [q, r] \rightarrow [p, r]$

$$\begin{bmatrix} p \\ p \end{bmatrix} \rightarrow \text{id}_{\mathcal{D}} \quad \text{and} \quad \begin{bmatrix} r \\ p \end{bmatrix} \rightarrow \begin{bmatrix} q \\ p \end{bmatrix} \circ \begin{bmatrix} r \\ q \end{bmatrix}$$

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- 3 Comonads as categories**
 - Pra-functors between copresheaf categories
 - Pra-comonads are categories
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Pra-functors between copresheaf categories

A *parametric right adjoint (pra)functor* $p: \mathbf{Set}^{\mathcal{C}} \rightarrow \mathbf{Set}^{\mathcal{D}}$ is

- one such that $p/1: \mathbf{Set}^{\mathcal{C}}/1 \rightarrow \mathbf{Set}^{\mathcal{D}}/p(1)$ is a right adjoint, or *equivalently*
- a connected-limit preserving functor, or *equivalently*
- a functor from $\mathcal{D}$ to the coproduct completion of $(\mathbf{Set}^{\mathcal{C}})^{\text{op}}$
 - Given $(I, X): \mathcal{D} \rightarrow \mathbf{Fam}((\mathbf{Set}^{\mathcal{C}})^{\text{op}})$, use $p(X')(d) := \sum_{i \in I_d} \mathbf{Set}^{\mathcal{C}}(X_i, X')$

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Suppose $p: \mathbf{Set}^{\mathcal{C}} \rightarrow \mathbf{Set}^{\mathcal{D}}$ is a prafunctor, i.e. $\mathcal{D} \xrightarrow{p} \mathbf{Fam}((\mathbf{Set}^{\mathcal{C}})^{\text{op}})$

- For any $\mathbf{Set}^{\mathcal{C}} \xrightarrow{q} \mathbf{Set}^{\mathcal{C}'}$, the left Kan extension $\begin{bmatrix} q \\ p \end{bmatrix}$ is a prafunctor:

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 \text{any } q \downarrow & \Downarrow & \uparrow \begin{bmatrix} q \\ p \end{bmatrix} \text{ pra} \\
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- Indeed, compose $\mathcal{D} \rightarrow \mathbf{Fam}((\mathbf{Set}^{\mathcal{C}})^{\text{op}}) \rightarrow \mathbf{Fam}((\mathbf{Set}^{\mathcal{C}'})^{\text{op}})$.

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A few more facts:

- Any right adjoint $\mathbf{Set}^{\mathcal{C}} \rightarrow \mathbf{Set}^{\mathcal{D}}$ is pra.
- elts: $\mathbf{Set}^{\mathcal{C}} \rightarrow \mathbf{Set}$ is pra, and has a right adjoint $\mathbf{Set} \rightarrow \mathbf{Set}^{\mathcal{C}}$, namely
- $X \mapsto C \mapsto \mathbf{Set}(H_C, X)$, where $H_C = \sum_{C' \in \text{Ob } \mathcal{C}} \mathcal{C}(C, C') = \text{elts}(\mathcal{Y}^{\mathcal{C}})$, “maps out”

Pra-comonads are categories

For any categories $\mathcal{C}, \mathcal{D}$, let $\mathbb{P}(\mathcal{C}, \mathcal{D}) := \mathbf{Pra}(\mathbf{Set}^{\mathcal{D}}, \mathbf{Set}^{\mathcal{C}})$.

- **Poly** $:= \mathbb{P}(1, 1)$. (And Gambino-Kock's $\mathbf{PolyFun}_{\mathbf{Set}} = \mathbb{P}|_{discCat}$.)
- Limit compl'n LC distributes over **Fam**, and Kleisli is $\mathbb{P} \cong \mathbf{Cat}_{\mathbf{Fam} \circ \mathbf{LC}}$.

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Categories = Polynomial comonads = Comonads in $\mathbb{P}(1, 1)$.

- Given $\mathcal{D}$, the comp'ite $\mathbf{Set} \xrightarrow{H} \mathbf{Set}^{\mathcal{D}} \xrightarrow{\text{elts}} \mathbf{Set}$ is a comonad d on $\mathbf{Set}$.
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- You can read off the objects and morphisms from the poly'l form.

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More generally, a comonad $\mathcal{D} \xrightarrow{c} \mathcal{D}$ in $\mathbb{P}$ has an underlying category.

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- Hence we get a category from any comonad c in $\mathbb{P}$.
- The counit $c \rightarrow \text{id}_{\mathcal{D}}$ gives a poly comonad map $elts \circ c \circ H \rightarrow d$
- Maps of poly'l comonads correspond to *cofunctors* between cat'ies.

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So let's find some comonads in $\mathbb{P}$.

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 - Density comonads
 - Finding density comonads
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Density comonads

For any $p: \mathbf{Set}^{\mathcal{C}} \rightarrow \mathbf{Set}^{\mathcal{D}}$, the left Kan extension $\begin{bmatrix} p \\ p \end{bmatrix}$ is a comonad.

- For closures, $[p, p]$ is a monoid by eval: $[p, p] \otimes [p, p] \rightarrow [p, p]$.
- Similarly, using coeval $\begin{bmatrix} r \\ p \end{bmatrix} \rightarrow \begin{bmatrix} q \\ p \end{bmatrix} \circ \begin{bmatrix} r \\ q \end{bmatrix}$ gives $\begin{bmatrix} p \\ p \end{bmatrix} \rightarrow \begin{bmatrix} p \\ p \end{bmatrix} \circ \begin{bmatrix} p \\ p \end{bmatrix}$.
- $\begin{bmatrix} p \\ p \end{bmatrix}: \mathbf{Set}^{\mathcal{D}} \rightarrow \mathbf{Set}^{\mathcal{D}}$ is called the *density* comonad of p .

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For p pra, we said $\begin{bmatrix} p \\ p \end{bmatrix}$ can be id'd with a cat'y and a cofunctor $\begin{bmatrix} p \\ p \end{bmatrix} \rightarrow \mathcal{D}$.

- That cofunctor is the identity iff p is dense, hence the name.
- We won't care about that case. We want to construct new categories.

Examples:

- Let $list: \mathbf{Set} \rightarrow \mathbf{Set}$ be $\sum_{N:\mathbb{N}} y^N$. Then $\begin{bmatrix} list \\ list \end{bmatrix} \cong \mathbf{FinSet}^{\text{op}}$.
- The maybe monad $y + 1$ has $\begin{bmatrix} y+1 \\ y+1 \end{bmatrix} \cong \boxed{\bullet \longrightarrow \bullet}$, the walking arrow.

Finding density comonads

Theorem

Suppose given a diagram of the following form, where k, p are pra:

$$\begin{array}{ccccc}
 \mathbf{Set}^{\mathcal{C}} & \xrightarrow{k} & \mathbf{Set}^{\mathcal{C}} & \xrightarrow{p} & \mathbf{Set}^{\mathcal{D}} \\
 t \downarrow & & \Downarrow & & \nearrow \\
 \mathbf{Set}^{\mathcal{C}} & & & & \\
 p \downarrow & & & & \\
 \mathbf{Set}^{\mathcal{D}} & & & &
 \end{array}
 \quad \begin{array}{c} \\ \\ [p \circ t] \\ [p \circ k] \end{array}$$

If k is a comonad and t is a monad and there is a distributive law $t \circ k \rightarrow k \circ t$ then the Kan extension $\begin{bmatrix} p \circ t \\ p \circ k \end{bmatrix}$ is a comonad.

E.g., if $k = \text{id}$ or $t = \text{id}$ then unique dist. law, so $\begin{bmatrix} p \circ t \\ p \end{bmatrix}$ & $\begin{bmatrix} p \\ p \circ k \end{bmatrix}$ comonads.

- Why is $\begin{bmatrix} p \circ t \\ p \circ k \end{bmatrix}$ a comonad? Factor the non-id as $\mathbf{Set}^{\mathcal{C}} \rightleftharpoons \mathbf{EM} \rightarrow \mathbf{Set}^{\mathcal{C}} \dots$
- ... and use $\begin{bmatrix} p \\ p \circ \text{lor} \end{bmatrix} = \begin{bmatrix} p \circ \text{el} \\ p \circ \text{el} \end{bmatrix}$ or $\begin{bmatrix} p \circ \text{rol} \\ p \end{bmatrix} = \begin{bmatrix} p \circ \text{or} \\ p \circ \text{or} \end{bmatrix}$, respectively.

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- 1 Introduction
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- 3 Comonads as categories
- 4 Categories by Kan extension
- 5 **Examples**
 - Lawvere theories
 - Arbitrary product completions
 - Nerves
- 6 Conclusion

Lawvere theories

Much of the rest is joint work with Brandon Shapiro.³

Let t be a finitary monad on **Set**. It has a Lawvere theory **Law** $_t$.

- Product-preserving functors **Law** $_t \rightarrow \mathbf{Set}$ are t -algebras.
- Mouthful: **Law** $_t$ is the opposite of the full subcategory of the Kleisli category **Set** $_t$ spanned by the finite sets.

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- Easier: **Law** $_t \cong \begin{bmatrix} list \circ t \\ list \end{bmatrix}$. Shown above: this is poly'l, even if t is not.
- Example: **Law** $_{id} \cong \mathbf{FinSet}^{op} \cong \begin{bmatrix} list \\ list \end{bmatrix}$.
- If t instead is κ -ary & S -sorted, let $\ell := \sum_{|A| < \kappa} y^A$. **Law** $_t = \begin{bmatrix} \ell \circ Sy \circ t \\ \ell \circ Sy \end{bmatrix}$.

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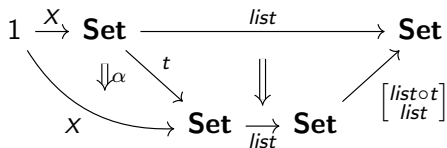
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Given a t -algebra $\alpha: t \circ X \rightarrow X$, what's the connection?



$list \circ X$ is a $\left[\begin{smallmatrix} list \circ t \\ list \end{smallmatrix} \right]$ -coalgebra
i.e. a **Law** _{t} -copsheaf
with carrier $list(X)$.

- i.e. a copsheaf ($N \mapsto X^N$): **Law** _{t} $\rightarrow$ **Set**. Its elements: $list(X)$.

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Arbitrary product completions

The product completion of a category is dual[?] to Lawvere construction.

- Let $\mathcal{C}$ be a category and c the corresponding polynomial comonad.
- Then $\left[\begin{smallmatrix} list \\ list \circ c \end{smallmatrix} \right]$ is the finite product completion of c .
- It's kinda dual to Lawvere theory: $\left[\begin{smallmatrix} list \circ t \\ list \end{smallmatrix} \right]$ vs. $\left[\begin{smallmatrix} list \\ list \circ c \end{smallmatrix} \right]$.

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You can ask for whatever “products” you want.

- $\left[\begin{smallmatrix} y^3 + y^7 \\ (y^3 + y^7) \circ c \end{smallmatrix} \right]$ is cat'y of 3-ary and 7-ary products in $\mathcal{C}$.
- This sounds weird, but the construction is at least well-behaved.

There's a normal lax double functor $\left[\begin{smallmatrix} p \\ p \circ c \end{smallmatrix} \right] : \mathbb{C}odisc(\mathbf{Poly}^{cart}) \times \mathbb{P}rof \rightarrow \mathbb{P}rof$.

- Here $\mathbb{C}odisc(\mathbf{Poly}^{cart})$ has tight $:= \mathbf{Poly}^{cart}$, loose $:=$ codiscrete.

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- Given $p \xrightarrow{cart} q$ and functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$, get functor $\left[\begin{smallmatrix} p \\ p \circ c \end{smallmatrix} \right] \rightarrow \left[\begin{smallmatrix} q \\ q \circ d \end{smallmatrix} \right]$. (Preserves ff in F)
- For each p, q and profunctor $\mathcal{C} \rightarrow \mathcal{D}$, get profunctor $\left[\begin{smallmatrix} p \\ p \circ c \end{smallmatrix} \right] \rightarrow \left[\begin{smallmatrix} q \\ q \circ d \end{smallmatrix} \right]$.

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- In fact, the action is lax $\circ$ -monoidal: $\left[\begin{smallmatrix} p \\ p \circ \left[\begin{smallmatrix} q \\ q \circ - \end{smallmatrix} \end{smallmatrix} \right] \end{smallmatrix} \right] \longrightarrow \left[\begin{smallmatrix} p \circ q \\ p \circ q \circ - \end{smallmatrix} \right]$.
- So since *list* is a cartesian monad, so is product completion $\left[\begin{smallmatrix} list \\ list \circ - \end{smallmatrix} \right]$.

Nerves

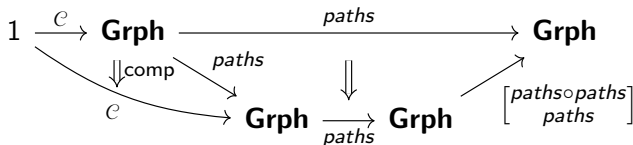
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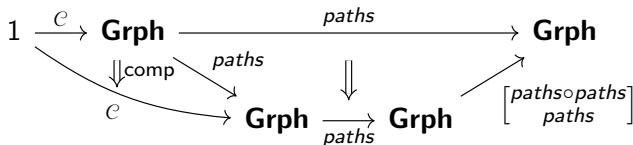
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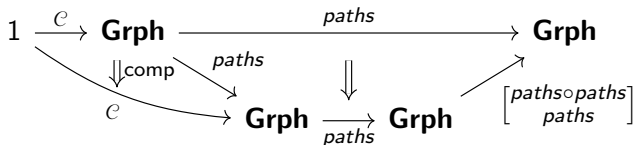


- This makes $\text{paths}(\mathcal{C})$ a coalgebra, i.e. a copresheaf, on $\begin{bmatrix} \text{paths} \circ \text{paths} \\ \text{paths} \end{bmatrix}$.
- And indeed, $\begin{bmatrix} \text{paths} \circ \text{paths} \\ \text{paths} \end{bmatrix} \cong \Delta^{\text{op}}$. Its copresheaves are simplicial sets.
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Same idea works for higher cat'l structures like nerves of double cat'ies.

- Pick *cell-shape* category $\mathcal{G}$, *compositions* pra-monad $t: \mathbf{Set}^{\mathcal{G}} \rightarrow \mathbf{Set}^{\mathcal{G}}$
- Nerves of that sort of thing are $\left[\begin{smallmatrix} t \circ t \\ t \end{smallmatrix} \right]$ -coalgebras, i.e. copresheaves.

Outline

- 1 Introduction
- 2 Basic theory of left Kan extensions
- 3 Comonads as categories
- 4 Categories by Kan extension
- 5 Examples
- 6 Conclusion**
 - Summary

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Thinking about categories as comonads opens up a new source of cat'ies.

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Lawvere theories, nerve theories, and finite product compl'ns arise this way.

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- Every algebra (monoidal double cat'y) is a Θ_t -presheaf.
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Quick plug: “Comonads on **Set** are spaces!” See A. Fairbanks's talk.

Thank you for your time; questions and comments welcome.

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