Categories by Kan extension

David I. Spivak

Topos Institute Berkeley Seminar
2026 September 08

Outline

Categories by Kan extension?

The left Kan extension of p along q is a *functor*. How can it be a *category*?

- Answer: It can be a comonad, e.g. a *density comonad*, and
- ... certain comonads c can be id'd with cat'ies $\mathcal{C}$,[1] with $\mathbf{Set}^c \cong \mathbf{Set}^{\mathcal{C}}$.
- I'll explain both parts of this story in the talk.

[1] Ahman and Uustalu (2016) *Directed Containers as Categories*, EPTCS 207.

Categories by Kan extension?

The left Kan extension of p along q is a *functor*. How can it be a *category*?

- Answer: It can be a comonad, e.g. a *density comonad*, and
- ... certain comonads c can be id'd with cat'ies $\mathcal{C}$,[1] with $\mathbf{Set}^c \cong \mathbf{Set}^{\mathcal{C}}$.
- I'll explain both parts of this story in the talk.

Application: nerve constructions and Lawvere theories (jww B. Shapiro).

- Example: let T be the free category monad on graphs.
- Then $\mathrm{Lan}_{T \circ T} T$ "is" Δ^{op}, whose copresheaves are simplicial sets.
- For any T-algebra $\mathcal{C}$, i.e. category, its nerve is induced by the UP.

[1] Ahman and Uustalu (2016) *Directed Containers as Categories*, EPTCS 207.

Plan of the talk

The plan of the talk is as follows:

- Recall basic theory of left Kan extensions and intro. $\begin{bmatrix} q \\ p \end{bmatrix}$ notation.
- Explain the "comonads as categories" idea.
- Give the "categories by Kan extension" construction.
- Discuss several examples, e.g. nerves (jww Brandon Shapiro)

Plan of the talk

The plan of the talk is as follows:

- Recall basic theory of left Kan extensions and intro. $\begin{bmatrix} q \\ p \end{bmatrix}$ notation.
- Explain the "comonads as categories" idea.
- Give the "categories by Kan extension" construction.
- Discuss several examples, e.g. nerves (jww Brandon Shapiro)

To keep statements simple, we'll elide mention of accessibility[2] in the talk.

[2] An object $c \in \mathcal{C}$ is *κ-small* if $\mathcal{C}(c, -)$ preserves κ-filtered colimits. A category is *accessible* if it has κ-filtered colimits and is generated under them by κ-small objects, for some regular cardinal κ. A functor is acc'ble if it preserves κ-filtered colimits for some κ.

Outline

Left Kan extensions

Given functors $q\colon \mathcal{C} \to \mathcal{C}'$ and $p\colon \mathcal{C} \to \mathcal{D}$, with $\mathcal{D}$ cocomplete, write

$$\begin{tikzcd} \mathcal{C} \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, phantom, "\Downarrow"] & \mathcal{D} \\ \mathcal{C}' \arrow[ur, bend right, "{\begin{bmatrix} q \\ p \end{bmatrix}}"'] & \end{tikzcd} \qquad \begin{bmatrix} q \\ p \end{bmatrix} := \mathrm{Lan}_q(p).$$

It is the left adjoint to precomposition by q. For $r\colon \mathcal{C}' \to \mathcal{D}$,

$$\mathrm{Hom}\left(\begin{bmatrix} q \\ p \end{bmatrix}, r \right) \cong \mathrm{Hom}(p, r \circ q).$$

It comes with unit $p \Rightarrow \begin{bmatrix} q \\ p \end{bmatrix} \circ q$ and counit $\begin{bmatrix} q \\ r \circ q \end{bmatrix} \Rightarrow r$.

Left Kan calculus

Intuitively, $\begin{bmatrix} q \\ p \end{bmatrix}$ acts like inner hom. Here are some examples:

- Variance: $P \to P'$ and $Q \leftarrow Q'$ induce $[Q, P] \to [Q', P']$

$$\begin{matrix} q & \longleftarrow & q' \\ p & \longrightarrow & p' \end{matrix} \qquad \rightsquigarrow \qquad \begin{bmatrix} q \\ p \end{bmatrix} \to \begin{bmatrix} q' \\ p' \end{bmatrix}$$

- Unit and tensor: $P \cong [I, P]$ and $[Q', [Q, P]] \cong [Q' \otimes Q, P]$.

$$p \cong \begin{bmatrix} \mathrm{id}_{\mathcal{C}} \\ p \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} q' \\ \begin{bmatrix} q \\ p \end{bmatrix} \end{bmatrix} \cong \begin{bmatrix} q' \circ q \\ p \end{bmatrix}.$$

- Duals: if $\ell \dashv r$ then: e.g. duals in a CCC: $[P \otimes V, Q] \cong [P, Q \otimes V^*]$.

$$\begin{bmatrix} q \\ p \circ r \end{bmatrix} \cong \begin{bmatrix} q \circ \ell \\ p \end{bmatrix}.$$

- Cocomposition: $I \to [p, p]$ and $[p, q] \otimes [q, r] \to [p, r]$

$$\begin{bmatrix} p \\ p \end{bmatrix} \to \mathrm{id}_{\mathcal{D}} \qquad \text{and} \qquad \begin{bmatrix} r \\ p \end{bmatrix} \to \begin{bmatrix} q \\ p \end{bmatrix} \circ \begin{bmatrix} r \\ q \end{bmatrix}$$

Outline

Pra-functors between copresheaf categories

A *parametric right adjoint (pra)functor* $p\colon \mathbf{Set}^{\mathcal{C}} \to \mathbf{Set}^{\mathcal{D}}$ is[3]

- one such that $p/1\colon \mathbf{Set}^{\mathcal{C}}/1 \to \mathbf{Set}^{\mathcal{D}}/p(1)$ is a right adjoint, or *equivalently*
- a connected-limit preserving functor, or *equivalently*
- a functor from $\mathcal{D}$ to the coproduct completion of $(\mathbf{Set}^{\mathcal{C}})^{\mathsf{op}}$.

We can write every such $\mathcal{D} \to \mathbf{Fam}\big((\mathbf{Set}^{\mathcal{C}})^{\mathsf{op}}\big)$ in polynomial-shaped form.

- Let's first write $(I, X)\colon \mathcal{D} \to \mathbf{Fam}\big((\mathbf{Set}^{\mathcal{C}})^{\mathsf{op}}\big)$, where
  - $I\colon \mathcal{D} \to \mathbf{Set}$ is the (per d) indexing set, and
  - $X\colon \prod_{d:\mathcal{D}} \prod_{i:I_d} \mathbf{Set}^{\mathcal{C}}$ is the (per d, per i) $\mathcal{C}$-set.

[3] These are also called familial functors.

Pra-functors between copresheaf categories

A *parametric right adjoint (pra)functor* $p\colon \mathbf{Set}^{\mathcal{C}} \to \mathbf{Set}^{\mathcal{D}}$ is[3]

- one such that $p/1\colon \mathbf{Set}^{\mathcal{C}}/1 \to \mathbf{Set}^{\mathcal{D}}/p(1)$ is a right adjoint, or *equivalently*
- a connected-limit preserving functor, or *equivalently*
- a functor from $\mathcal{D}$ to the coproduct completion of $(\mathbf{Set}^{\mathcal{C}})^{\mathsf{op}}$.

We can write every such $\mathcal{D} \to \mathbf{Fam}\big((\mathbf{Set}^{\mathcal{C}})^{\mathsf{op}}\big)$ in polynomial-shaped form.

- Let's first write $(I, X)\colon \mathcal{D} \to \mathbf{Fam}\big((\mathbf{Set}^{\mathcal{C}})^{\mathsf{op}}\big)$, where
  - $I\colon \mathcal{D} \to \mathbf{Set}$ is the (per d) indexing set, and
  - $X\colon \prod_{d:\mathcal{D}} \prod_{i:I_d} \mathbf{Set}^{\mathcal{C}}$ is the (per d, per i) $\mathcal{C}$-set.
- Then the functor $p\colon \mathbf{Set}^{\mathcal{C}} \to \mathbf{Set}^{\mathcal{D}}$ is given by:

$$p(X')(d) \coloneqq \sum_{i \in I_d} \mathbf{Set}^{\mathcal{C}}(X_i, X')$$

i.e. $p_d \coloneqq \sum_{i \in I_d} y^{X_i}$, s.t. $d \to d'$ functorially induces n.t. $p_d \to p_{d'}$.

[3] These are also called familial functors.

Pra-functors between copresheaf categories

A *parametric right adjoint (pra)functor* $p\colon \mathbf{Set}^{\mathcal{C}} \to \mathbf{Set}^{\mathcal{D}}$ is[3]

- one such that $p/1\colon \mathbf{Set}^{\mathcal{C}}/1 \to \mathbf{Set}^{\mathcal{D}}/p(1)$ is a right adjoint, or *equivalently*
- a connected-limit preserving functor, or *equivalently*
- a functor from $\mathcal{D}$ to the coproduct completion of $(\mathbf{Set}^{\mathcal{C}})^{\mathrm{op}}$.

We can write every such $\mathcal{D} \to \mathbf{Fam}\big((\mathbf{Set}^{\mathcal{C}})^{\mathrm{op}}\big)$ in polynomial-shaped form.

- Let's first write $(I, X)\colon \mathcal{D} \to \mathbf{Fam}\big((\mathbf{Set}^{\mathcal{C}})^{\mathrm{op}}\big)$, where
  - $I\colon \mathcal{D} \to \mathbf{Set}$ is the (per d) indexing set, and
  - $X\colon \prod_{d:\mathcal{D}} \prod_{i:I_d} \mathbf{Set}^{\mathcal{C}}$ is the (per d, per i) $\mathcal{C}$-set.
- Then the functor $p\colon \mathbf{Set}^{\mathcal{C}} \to \mathbf{Set}^{\mathcal{D}}$ is given by:

$$p(X')(d) \coloneqq \sum_{i \in I_d} \mathbf{Set}^{\mathcal{C}}(X_i, X')$$

i.e. $p_d \coloneqq \sum_{i \in I_d} y^{X_i}$, s.t. $d \to d'$ functorially induces n.t. $p_d \to p_{d'}$.

The rest of the talk will be about prafunctors $\mathbf{Set}^{\mathcal{C}} \to \mathbf{Set}^{\mathcal{D}}$ so... questions?

[3] These are also called familial functors.

Left Kan extension along anything preserves pra

Suppose $p\colon \mathbf{Set}^{\mathscr{C}} \to \mathbf{Set}^{\mathscr{D}}$ is a prafunctor (i.e. preserves connected limits).

- For *any* $\mathbf{Set}^{\mathscr{C}} \xrightarrow{q} \mathbf{Set}^{\mathscr{C}'}$, the left Kan extension $\begin{bmatrix} q \\ p \end{bmatrix}$ is a prafunctor:

$$\begin{tikzcd}
\mathbf{Set}^{\mathscr{C}} \arrow[rr, "\overset{\text{pra}}{p}"] \arrow[d, "\text{any } q"'] & \arrow[d, phantom, "\Downarrow"] & \mathbf{Set}^{\mathscr{D}} \\
\mathbf{Set}^{\mathscr{C}'} \arrow[urr, bend right, "{\begin{bmatrix} q \\ p \end{bmatrix} \text{ pra}}"'] & &
\end{tikzcd}$$

- It's the composite $\mathscr{D} \to \mathbf{Fam}\big((\mathbf{Set}^{\mathscr{C}})^{\mathrm{op}}\big) \xrightarrow{\mathbf{Fam}(q^{\mathrm{op}})} \mathbf{Fam}\big((\mathbf{Set}^{\mathscr{C}'})^{\mathrm{op}}\big)$.

Left Kan extension along anything preserves pra

Suppose $p\colon \mathbf{Set}^{\mathcal{C}} \to \mathbf{Set}^{\mathcal{D}}$ is a prafunctor (i.e. preserves connected limits).

- For *any* $\mathbf{Set}^{\mathcal{C}} \xrightarrow{q} \mathbf{Set}^{\mathcal{C}'}$, the left Kan extension $\begin{bmatrix} q \\ p \end{bmatrix}$ is a prafunctor:

$$\begin{tikzcd}
\mathbf{Set}^{\mathcal{C}} \arrow[rr, "\overset{\text{pra}}{p}"] \arrow[d, "\text{any } q"'] & \arrow[d, phantom, "\Downarrow"] & \mathbf{Set}^{\mathcal{D}} \\
\mathbf{Set}^{\mathcal{C}'} \arrow[urr, bend right, "\begin{bmatrix} q \\ p \end{bmatrix} \text{ pra}"'] & &
\end{tikzcd}$$

- It's the composite $\mathcal{D} \to \mathbf{Fam}\big((\mathbf{Set}^{\mathcal{C}})^{\mathsf{op}}\big) \xrightarrow{\mathbf{Fam}(q^{\mathsf{op}})} \mathbf{Fam}\big((\mathbf{Set}^{\mathcal{C}'})^{\mathsf{op}}\big)$.

A few more facts:

- Any right adjoint $\mathbf{Set}^{\mathcal{C}} \to \mathbf{Set}^{\mathcal{D}}$ is pra.
- $\mathsf{elts}\colon \mathbf{Set}^{\mathcal{C}} \to \mathbf{Set}$ is pra, and has a right adjoint $\mathbf{Set} \xrightarrow{\mathcal{H}} \mathbf{Set}^{\mathcal{C}}$, namely
- $X \mapsto C \mapsto \mathbf{Set}(H_C, X)$, where $H_C = \sum_{C' \in \mathsf{Ob}\,\mathcal{C}} \mathcal{C}(C, C')$ is "maps out".
- In fact, $\begin{tikzcd} \mathbf{Set}^{\mathcal{C}} \arrow[r, shift left=2, "\mathsf{elts}"] \arrow[r, phantom, "\Rightarrow"] & \mathbf{Set} \arrow[l, shift left=2, "\mathcal{H}"] \end{tikzcd}$ is comonadic.

Pra-comonads are categories

For any categories $\mathcal{C}, \mathcal{D}$, let $\mathbb{P}(\mathcal{C}, \mathcal{D}) \coloneqq \mathbf{Pra}(\mathbf{Set}^{\mathcal{D}}, \mathbf{Set}^{\mathcal{C}})$.

- **Poly** $\coloneqq \mathbb{P}(1,1)$. (And Gambino-Kock's $\mathbb{P}\mathbf{olyFun}_{\mathbf{Set}} = \mathbb{P}\big|_{discCat}$.)
- Limit compl'n LC distributes over **Fam**, and Kleisli is $\mathbb{P} \cong \mathbb{C}\mathbf{at}_{\mathbf{Fam} \circ \mathsf{LC}}$.

Pra-comonads are categories

For any categories $\mathcal{C}, \mathcal{D}$, let $\mathbb{P}(\mathcal{C}, \mathcal{D}) \coloneqq \mathbf{Pra}(\mathbf{Set}^{\mathcal{D}}, \mathbf{Set}^{\mathcal{C}})$.

- **Poly** $\coloneqq \mathbb{P}(1,1)$. (And Gambino-Kock's $\mathbb{P}\mathbf{olyFun}_{\mathbf{Set}} = \mathbb{P}\big|_{discCat}$.)
- Limit compl'n LC distributes over **Fam**, and Kleisli is $\mathbb{P} \cong \mathbb{C}\mathbf{at}_{\mathbf{Fam} \circ \mathsf{LC}}$.

Categories = Polynomial comonads = Comonads in $\mathbb{P}(1,1)$.

- Given $\mathcal{D}$, the comp'ite $\mathbf{Set} \xrightarrow{\mathcal{H}} \mathbf{Set}^{\mathcal{D}} \xrightarrow{\text{elts}} \mathbf{Set}$ is a comonad d on **Set**.
- As a polynomial it is $\sum_{D \in \mathsf{Ob}\,\mathcal{D}} y^{H_D}$, where again H_D =maps out of D.
- You can read off the objects and morphisms from the poly'l form.

Pra-comonads are categories

For any categories $\mathcal{C}, \mathcal{D}$, let $\mathbb{P}(\mathcal{C}, \mathcal{D}) \coloneqq \mathbf{Pra}(\mathbf{Set}^{\mathcal{D}}, \mathbf{Set}^{\mathcal{C}})$.

- **Poly** $\coloneqq \mathbb{P}(1, 1)$. (And Gambino-Kock's $\mathbb{P}\mathbf{olyFun}_{\mathbf{Set}} = \mathbb{P}\big|_{discCat}$.)
- Limit compl'n LC distributes over **Fam**, and Kleisli is $\mathbb{P} \cong \mathbb{C}\mathbf{at}_{\mathbf{Fam} \circ \mathrm{LC}}$.

Categories $=$ Polynomial comonads $=$ Comonads in $\mathbb{P}(1, 1)$.

- Given $\mathcal{D}$, the comp'ite $\mathbf{Set} \xrightarrow{\mathcal{H}} \mathbf{Set}^{\mathcal{D}} \xrightarrow{\text{elts}} \mathbf{Set}$ is a comonad d on **Set**.
- As a polynomial it is $\sum_{D \in \mathrm{Ob}\,\mathcal{D}} y^{H_D}$, where again H_D =maps out of D.
- You can read off the objects and morphisms from the poly'l form.

More gen'ly, a comonad (c, ϵ, δ) with $c \in \mathbb{P}(\mathcal{D}, \mathcal{D})$ has an underlying cat'y.

- Conjugate $\mathbf{Set} \xrightarrow{\mathcal{H}} \mathbf{Set}^{\mathcal{D}} \xrightarrow{c} \mathbf{Set}^{\mathcal{D}} \xrightarrow{\text{elts}} \mathbf{Set}$ is a polynomial comonad.
- Hence we get a category from any comonad c in $\mathbb{P}$.
- The counit $c \to \mathrm{id}_{\mathcal{D}}$ gives a poly comonad map $(\text{elts} \circ c \circ \mathcal{H}) \to d$
- Maps of poly'l comonads correspond to *cofunctors* between cat'ies.
- $\mathbf{Set}^{\mathcal{C}} \cong c$-**Coalg**. For $\mathcal{C} \xrightarrow{X} \mathbf{Set}$, elements $\text{elts}(X)$ carry the coalgebra

Pra-comonads are categories

For any categories $\mathcal{C}, \mathcal{D}$, let $\mathbb{P}(\mathcal{C}, \mathcal{D}) := \mathbf{Pra}(\mathbf{Set}^{\mathcal{D}}, \mathbf{Set}^{\mathcal{C}})$.

- $\mathbf{Poly} := \mathbb{P}(1,1)$. (And Gambino-Kock's $\mathbb{P}\mathbf{olyFun}_{\mathbf{Set}} = \mathbb{P}\big|_{discCat}$.)
- Limit compl'n LC distributes over **Fam**, and Kleisli is $\mathbb{P} \cong \mathbb{C}\mathbf{at}_{\mathbf{Fam} \circ \mathsf{LC}}$.

Categories = Polynomial comonads = Comonads in $\mathbb{P}(1,1)$.

- Given $\mathcal{D}$, the comp'ite $\mathbf{Set} \xrightarrow{\mathcal{H}} \mathbf{Set}^{\mathcal{D}} \xrightarrow{\text{elts}} \mathbf{Set}$ is a comonad d on **Set**.
- As a polynomial it is $\sum_{D \in \mathrm{Ob}\,\mathcal{D}} y^{H_D}$, where again H_D =maps out of D.
- You can read off the objects and morphisms from the poly'l form.

More gen'ly, a comonad (c, ϵ, δ) with $c \in \mathbb{P}(\mathcal{D}, \mathcal{D})$ has an underlying cat'y.

- Conjugate $\mathbf{Set} \xrightarrow{\mathcal{H}} \mathbf{Set}^{\mathcal{D}} \xrightarrow{c} \mathbf{Set}^{\mathcal{D}} \xrightarrow{\text{elts}} \mathbf{Set}$ is a polynomial comonad.
- Hence we get a category from any comonad c in $\mathbb{P}$.
- The counit $c \to \mathrm{id}_{\mathcal{D}}$ gives a poly comonad map $(\text{elts} \circ c \circ \mathcal{H}) \to d$
- Maps of poly'l comonads correspond to *cofunctors* between cat'ies.
- $\mathbf{Set}^{\mathcal{C}} \cong c\text{-}\mathbf{Coalg}$. For $\mathcal{C} \xrightarrow{X} \mathbf{Set}$, elements $\text{elts}(X)$ carry the coalgebra

Next up: find a source of comonads in $\mathbb{P}$. Then: see them as categories.

Outline

Density comonads

For any $p\colon \mathbf{Set}^{\mathscr{C}} \to \mathbf{Set}^{\mathscr{D}}$, the left Kan extension $\begin{bmatrix} p \\ p \end{bmatrix}$ is a comonad.

- For closures, $[p, p]$ is a monoid by eval: $[p, p] \otimes [p, p] \to [p, p]$.
- Similarly, using coeval $\begin{bmatrix} r \\ p \end{bmatrix} \to \begin{bmatrix} q \\ p \end{bmatrix} \circ \begin{bmatrix} r \\ q \end{bmatrix}$ gives $\begin{bmatrix} p \\ p \end{bmatrix} \to \begin{bmatrix} p \\ p \end{bmatrix} \circ \begin{bmatrix} p \\ p \end{bmatrix}$.
- $\begin{bmatrix} p \\ p \end{bmatrix} : \mathbf{Set}^{\mathscr{D}} \to \mathbf{Set}^{\mathscr{D}}$ is called the *density* comonad of p.

Density comonads

For any $p \colon \mathbf{Set}^{\mathscr{C}} \to \mathbf{Set}^{\mathscr{D}}$, the left Kan extension $\begin{bmatrix} p \\ p \end{bmatrix}$ is a comonad.

- For closures, $[p, p]$ is a monoid by eval: $[p, p] \otimes [p, p] \to [p, p]$.
- Similarly, using coeval $\begin{bmatrix} r \\ p \end{bmatrix} \to \begin{bmatrix} q \\ p \end{bmatrix} \circ \begin{bmatrix} r \\ q \end{bmatrix}$ gives $\begin{bmatrix} p \\ p \end{bmatrix} \to \begin{bmatrix} p \\ p \end{bmatrix} \circ \begin{bmatrix} p \\ p \end{bmatrix}$.
- $\begin{bmatrix} p \\ p \end{bmatrix} \colon \mathbf{Set}^{\mathscr{D}} \to \mathbf{Set}^{\mathscr{D}}$ is called the *density* comonad of p.

For p pra, we said $\begin{bmatrix} p \\ p \end{bmatrix}$ can be id'd with a cat'y and a cofunctor $\begin{bmatrix} p \\ p \end{bmatrix} \to \mathscr{D}$.

- That cofunctor is the identity iff p is dense, hence the name.
- We won't care about that case. We want to construct new categories.

Density comonads

For any $p\colon \mathbf{Set}^{\mathscr{C}} \to \mathbf{Set}^{\mathscr{D}}$, the left Kan extension $\begin{bmatrix} p \\ p \end{bmatrix}$ is a comonad.

- For closures, $[p, p]$ is a monoid by eval: $[p, p] \otimes [p, p] \to [p, p]$.
- Similarly, using coeval $\begin{bmatrix} r \\ p \end{bmatrix} \to \begin{bmatrix} q \\ p \end{bmatrix} \circ \begin{bmatrix} r \\ q \end{bmatrix}$ gives $\begin{bmatrix} p \\ p \end{bmatrix} \to \begin{bmatrix} p \\ p \end{bmatrix} \circ \begin{bmatrix} p \\ p \end{bmatrix}$.
- $\begin{bmatrix} p \\ p \end{bmatrix} : \mathbf{Set}^{\mathscr{D}} \to \mathbf{Set}^{\mathscr{D}}$ is called the *density* comonad of p.

For p pra, we said $\begin{bmatrix} p \\ p \end{bmatrix}$ can be id'd with a cat'y and a cofunctor $\begin{bmatrix} p \\ p \end{bmatrix} \to \mathscr{D}$.

- That cofunctor is the identity iff p is dense, hence the name.
- We won't care about that case. We want to construct new categories.

Examples:

- Let *list*$\colon \mathbf{Set} \to \mathbf{Set}$ be $\sum_{N:\mathbb{N}} y^N$. Then $\begin{bmatrix} \mathit{list} \\ \mathit{list} \end{bmatrix} \cong \mathbf{FinSet}^{\mathsf{op}}$.
- The maybe monad $y + 1$ has $\begin{bmatrix} y+1 \\ y+1 \end{bmatrix} \cong \boxed{\bullet \longrightarrow \bullet}$, the walking arrow.

Finding density comonads

Theorem

Suppose given a diagram of the following form, where k, p are pra:

$$
\begin{tikzcd}
\mathbf{Set}^{\mathcal{C}} \arrow[r, "k"] \arrow[d, "t"'] & \mathbf{Set}^{\mathcal{C}} \arrow[r, "p"] \arrow[dd, phantom, "\Downarrow"] & \mathbf{Set}^{\mathcal{D}} \\
\mathbf{Set}^{\mathcal{C}} \arrow[d, "p"'] & & \\
\mathbf{Set}^{\mathcal{D}} \arrow[uurr, bend right, "{\begin{bmatrix} p\circ t \\ p\circ k \end{bmatrix}}"'] & &
\end{tikzcd}
$$

If k is a comonad and t is a monad and there is a distributive law $t \circ k \to k \circ t$ then the Kan extension $\begin{bmatrix} p\circ t \\ p\circ k \end{bmatrix}$ is a comonad.

Finding density comonads

Theorem

Suppose given a diagram of the following form, where k, p are pra:

$$\begin{tikzcd}
\mathbf{Set}^{\mathscr{C}} \arrow[r, "k"] \arrow[d, "t"'] & \mathbf{Set}^{\mathscr{C}} \arrow[r, "p"] \arrow[dd, phantom, "\Downarrow"] & \mathbf{Set}^{\mathscr{D}} \\
\mathbf{Set}^{\mathscr{C}} \arrow[d, "p"'] & & \\
\mathbf{Set}^{\mathscr{D}} \arrow[rruu, bend right, "{\begin{bmatrix} p\circ t \\ p\circ k \end{bmatrix}}"'] & &
\end{tikzcd}$$

If k is a comonad and t is a monad and there is a distributive law $t \circ k \to k \circ t$ then the Kan extension $\begin{bmatrix} p\circ t \\ p\circ k \end{bmatrix}$ is a comonad.

E.g., if $k = \mathrm{id}$ or $t = \mathrm{id}$ then unique dist. law, so $\begin{bmatrix} p\circ t \\ p \end{bmatrix}$ & $\begin{bmatrix} p \\ p\circ k \end{bmatrix}$ comonads.

- Why is $\begin{bmatrix} p\circ t \\ p\circ k \end{bmatrix}$ a comonad? Factor the non-id as $\mathbf{Set}^{\mathscr{C}} \leftrightarrows \mathbf{EM} \to \mathbf{Set}^{\mathscr{C}} \ldots$
- … and use $\begin{bmatrix} p \\ p\circ\ell\circ r \end{bmatrix} = \begin{bmatrix} p\circ\ell \\ p\circ\ell \end{bmatrix}$ or $\begin{bmatrix} p\circ r\circ\ell \\ p \end{bmatrix} = \begin{bmatrix} p\circ r \\ p\circ r \end{bmatrix}$, respectively.

Outline

Ex 1: Lawvere theories

Much of the rest is joint work with Brandon Shapiro.[4]

Let t be a finitary monad on **Set**. It has a Lawvere theory $\mathbf{Law}_t$.

- Product-preserving functors $\mathbf{Law}_t \to \mathbf{Set}$ are t-algebras.
- Mouthful: $\mathbf{Law}_t$ is the opposite of the full subcategory of the Kleisli category $\mathbf{Set}_t$ spanned by the finite sets.

[4] *A Polynomial Construction of Nerves for Higher Categories*. 2405.13157

Ex 1: Lawvere theories

Much of the rest is joint work with Brandon Shapiro.[4]

Let t be a finitary monad on **Set**. It has a Lawvere theory $\mathbf{Law}_t$.

- Product-preserving functors $\mathbf{Law}_t \to \mathbf{Set}$ are t-algebras.
- Mouthful: $\mathbf{Law}_t$ is the opposite of the full subcategory of the Kleisli category $\mathbf{Set}_t$ spanned by the finite sets.
- Easier: $\mathbf{Law}_t \cong \begin{bmatrix} \mathit{list} \circ t \\ \mathit{list} \end{bmatrix}$. Shown above: this is poly'l, even if t is not.
- Example: $\mathbf{Law}_{\mathrm{id}} \cong \mathbf{FinSet}^{\mathrm{op}} \cong \begin{bmatrix} \mathit{list} \\ \mathit{list} \end{bmatrix}$.
- If t instead is κ-ary & S-sorted, let $\ell := \sum_{|A|<\kappa} y^A$. $\mathbf{Law}_t = \begin{bmatrix} \ell \circ Sy \circ t \\ \ell \circ Sy \end{bmatrix}$.

[4] *A Polynomial Construction of Nerves for Higher Categories*. 2405.13157

Ex 1: Lawvere theories

Much of the rest is joint work with Brandon Shapiro.[4]

Let t be a finitary monad on **Set**. It has a Lawvere theory $\mathbf{Law}_t$.

- Product-preserving functors $\mathbf{Law}_t \to \mathbf{Set}$ are t-algebras.
- Mouthful: $\mathbf{Law}_t$ is the opposite of the full subcategory of the Kleisli category $\mathbf{Set}_t$ spanned by the finite sets.
- Easier: $\mathbf{Law}_t \cong \begin{bmatrix} \mathit{list} \circ t \\ \mathit{list} \end{bmatrix}$. Shown above: this is poly'l, even if t is not.
- Example: $\mathbf{Law}_{\mathrm{id}} \cong \mathbf{FinSet}^{\mathrm{op}} \cong \begin{bmatrix} \mathit{list} \\ \mathit{list} \end{bmatrix}$.
- If t instead is κ-ary & S-sorted, let $\ell := \sum_{|A|<\kappa} y^A$. $\mathbf{Law}_t = \begin{bmatrix} \ell \circ S y \circ t \\ \ell \circ S y \end{bmatrix}$.

Given a t-algebra $\alpha \colon t \circ X \to X$, what's the connection?

$$\begin{tikzcd}
1 \arrow[r, "X"] \arrow[drr, bend right, "X"'] & \mathbf{Set} \arrow[rrr, "\mathit{list}"] \arrow[dr, "t"] \arrow[d, phantom, "\Downarrow\alpha"] & & \arrow[d, phantom, "\Downarrow"] & \mathbf{Set} \\
& {} & \mathbf{Set} \arrow[r, "\mathit{list}"'] & \mathbf{Set} \arrow[ur, "\begin{bmatrix} \mathit{list} \circ t \\ \mathit{list} \end{bmatrix}"'] &
\end{tikzcd}$$

$\mathit{list} \circ X$ is a $\begin{bmatrix} \mathit{list} \circ t \\ \mathit{list} \end{bmatrix}$-coalgebra i.e. a $\mathbf{Law}_t$-copresheaf with carrier $\mathit{list}(X)$.

- i.e. a copresheaf $(N \mapsto X^N) \colon \mathbf{Law}_t \to \mathbf{Set}$. Its elements: $\mathit{list}(X)$.

[4] *A Polynomial Construction of Nerves for Higher Categories*. 2405.13157

Ex 2: Arbitrary product completions

The product completion of a category is dual? to Lawvere construction.

- Let $\mathcal{C}$ be a category and c the corresponding polynomial comonad.
- Then $\begin{bmatrix} \mathit{list} \\ \mathit{list} \circ c \end{bmatrix}$ is the finite product completion of c.
- It's kinda dual to Lawvere theory: $\begin{bmatrix} \mathit{list} \circ t \\ \mathit{list} \end{bmatrix}$ vs. $\begin{bmatrix} \mathit{list} \\ \mathit{list} \circ c \end{bmatrix}$.

Ex 2: Arbitrary product completions

The product completion of a category is dual? to Lawvere construction.

- Let $\mathcal{C}$ be a category and c the corresponding polynomial comonad.
- Then $\begin{bmatrix} \mathit{list} \\ \mathit{list} \circ c \end{bmatrix}$ is the finite product completion of c.
- It's kinda dual to Lawvere theory: $\begin{bmatrix} \mathit{list} \circ t \\ \mathit{list} \end{bmatrix}$ vs. $\begin{bmatrix} \mathit{list} \\ \mathit{list} \circ c \end{bmatrix}$.

You can ask for whatever "products" you want.

- $\begin{bmatrix} y^3 + y^7 \\ (y^3 + y^7) \circ c \end{bmatrix}$ is cat'y of 3-ary and 7-ary products in $\mathcal{C}$.
- This sounds weird, but the construction is at least well-behaved.

There's a normal lax double functor $\begin{bmatrix} p \\ p \circ c \end{bmatrix} : \mathbb{C}\mathbf{odisc}(\mathbf{Poly}^{\mathbf{cart}}) \times \mathbb{P}\mathbf{rof} \to \mathbb{P}\mathbf{rof}$.

- Here $\mathbb{C}\mathbf{odisc}(\mathbf{Poly}^{\mathbf{cart}})$ has tight $:= \mathbf{Poly}^{\mathbf{cart}}$, loose $:=$ codiscrete.

Ex 2: Arbitrary product completions

The product completion of a category is dual$^?$ to Lawvere construction.

- Let $\mathcal{C}$ be a category and c the corresponding polynomial comonad.
- Then $\begin{bmatrix} \mathit{list} \\ \mathit{list} \circ c \end{bmatrix}$ is the finite product completion of c.
- It's kinda dual to Lawvere theory: $\begin{bmatrix} \mathit{list} \circ t \\ \mathit{list} \end{bmatrix}$ vs. $\begin{bmatrix} \mathit{list} \\ \mathit{list} \circ c \end{bmatrix}$.

You can ask for whatever "products" you want.

- $\begin{bmatrix} y^3+y^7 \\ (y^3+y^7) \circ c \end{bmatrix}$ is cat'y of 3-ary and 7-ary products in $\mathcal{C}$.
- This sounds weird, but the construction is at least well-behaved.

There's a normal lax double functor $\begin{bmatrix} p \\ p \circ c \end{bmatrix} : \mathbb{C}\mathbf{odisc}(\mathbf{Poly}^{\mathbf{cart}}) \times \mathbb{P}\mathbf{rof} \to \mathbb{P}\mathbf{rof}$.

- Here $\mathbb{C}\mathbf{odisc}(\mathbf{Poly}^{\mathbf{cart}})$ has tight $:= \mathbf{Poly}^{\mathbf{cart}}$, loose $:=$ codiscrete.
- Given $p \xrightarrow{\text{cart}} q$ and functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$, get functor $\begin{bmatrix} p \\ p \circ c \end{bmatrix} \to \begin{bmatrix} q \\ q \circ d \end{bmatrix}$. (Preserves ff in F)
- For each p, q and profunctor $\mathcal{C} \nrightarrow \mathcal{D}$, get profunctor $\begin{bmatrix} p \\ p \circ c \end{bmatrix} \nrightarrow \begin{bmatrix} q \\ q \circ d \end{bmatrix}$.

Ex 2: Arbitrary product completions

The product completion of a category is dual$^?$ to Lawvere construction.

- Let $\mathcal{C}$ be a category and c the corresponding polynomial comonad.
- Then $\begin{bmatrix} list \\ list \circ c \end{bmatrix}$ is the finite product completion of c.
- It's kinda dual to Lawvere theory: $\begin{bmatrix} list \circ t \\ list \end{bmatrix}$ vs. $\begin{bmatrix} list \\ list \circ c \end{bmatrix}$.

You can ask for whatever "products" you want.

- $\begin{bmatrix} y^3+y^7 \\ (y^3+y^7)\circ c \end{bmatrix}$ is cat'y of 3-ary and 7-ary products in $\mathcal{C}$.
- This sounds weird, but the construction is at least well-behaved.

There's a normal lax double functor $\begin{bmatrix} p \\ p \circ c \end{bmatrix}$: $\mathbb{C}\mathbf{odisc}(\mathbf{Poly}^{\mathbf{cart}}) \times \mathbb{P}\mathbf{rof} \to \mathbb{P}\mathbf{rof}$.

- Here $\mathbb{C}\mathbf{odisc}(\mathbf{Poly}^{\mathbf{cart}})$ has tight $:= \mathbf{Poly}^{\mathbf{cart}}$, loose $:=$ codiscrete.
- Given $p \xrightarrow{\text{cart}} q$ and functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$, get functor $\begin{bmatrix} p \\ p \circ c \end{bmatrix} \to \begin{bmatrix} q \\ q \circ d \end{bmatrix}$. (Preserves ff in F)
- For each p, q and profunctor $\mathcal{C} \nrightarrow \mathcal{D}$, get profunctor $\begin{bmatrix} p \\ p \circ c \end{bmatrix} \nrightarrow \begin{bmatrix} q \\ q \circ d \end{bmatrix}$.
- In fact, the action is lax $\circ$-monoidal: $\begin{bmatrix} p \\ p \circ \begin{bmatrix} q \\ q \circ - \end{bmatrix} \end{bmatrix} \longrightarrow \begin{bmatrix} p \circ q \\ p \circ q \circ - \end{bmatrix}$.
- So since *list* is a cartesian monad, so is product completion $\begin{bmatrix} list \\ list \circ - \end{bmatrix}$.

Ex 3: Nerves

The nerve of a category is a copresheaf on Δ^{op}. Let's find Δ^{op}.

- A category is a graph for which each path of edges is assigned an edge
- That is $\mathcal{C}$ is an algebra for the *paths* monad on **Grph** $\cong$ $\mathbf{Set}^{\bullet\rightrightarrows\bullet}$.

Ex 3: Nerves

The nerve of a category is a copresheaf on Δ^{op}. Let's find Δ^{op}.

- A category is a graph for which each path of edges is assigned an edge
- That is $\mathcal{C}$ is an algebra for the *paths* monad on **Grph** $\cong$ $\mathbf{Set}^{\bullet\rightrightarrows\bullet}$.
- We have comp: $paths(\mathcal{C}) \to \mathcal{C}$. Just like for Lawvere theories:

$$\begin{tikzcd}
1 \arrow[r, "\mathcal{C}"] \arrow[drr, bend right, "\mathcal{C}"'] & \mathbf{Grph} \arrow[rrr, "paths"] \arrow[dr, "paths"] \arrow[d, phantom, "\Downarrow\mathrm{comp}"] & & & \mathbf{Grph} \\
& {} & \mathbf{Grph} \arrow[r, "paths"'] \arrow[ur, phantom, "\Downarrow"] & \mathbf{Grph} \arrow[ur, "\begin{bmatrix} paths\circ paths \\ paths \end{bmatrix}"'] &
\end{tikzcd}$$

Ex 3: Nerves

The nerve of a category is a copresheaf on Δ^{op}. Let's find Δ^{op}.

- A category is a graph for which each path of edges is assigned an edge
- That is $\mathcal{C}$ is an algebra for the *paths* monad on $\mathbf{Grph} \cong \mathbf{Set}^{\bullet \rightrightarrows \bullet}$.
- We have comp: $\mathit{paths}(\mathcal{C}) \to \mathcal{C}$. Just like for Lawvere theories:

$$\begin{tikzcd}
1 \arrow[r, "\mathcal{C}"] \arrow[drr, bend right, "\mathcal{C}"'] & \mathbf{Grph} \arrow[rrr, "\mathit{paths}"] \arrow[dr, "\mathit{paths}"] \arrow[d, phantom, "\Downarrow \mathrm{comp}"] & & & \mathbf{Grph} \\
& {} & \mathbf{Grph} \arrow[r, "\mathit{paths}"'] & \mathbf{Grph} \arrow[ur, "{\left[\begin{smallmatrix} \mathit{paths}\circ \mathit{paths} \\ \mathit{paths} \end{smallmatrix}\right]}"'] \arrow[u, phantom, "\Downarrow"] &
\end{tikzcd}$$

- This makes $\mathit{paths}(\mathcal{C})$ a coalgebra, i.e. a copresheaf, on $\left[\begin{smallmatrix} \mathit{paths}\circ \mathit{paths} \\ \mathit{paths} \end{smallmatrix}\right]$.
- And indeed, $\left[\begin{smallmatrix} \mathit{paths}\circ \mathit{paths} \\ \mathit{paths} \end{smallmatrix}\right] \cong \Delta^{\mathrm{op}}$. Its copresheaves are simplicial sets.
- Also $\left[\begin{smallmatrix} \mathit{paths} \\ \mathit{paths} \end{smallmatrix}\right] \cong \Delta^{\mathrm{op}}_{\mathrm{inert}}$, and the unit induces $\left[\begin{smallmatrix} \mathit{paths} \\ \mathit{paths} \end{smallmatrix}\right] \xrightarrow{\mathrm{io}} \left[\begin{smallmatrix} \mathit{paths}\circ \mathit{paths} \\ \mathit{paths} \end{smallmatrix}\right]$

Ex 3: Nerves

The nerve of a category is a copresheaf on Δ^{op}. Let's find Δ^{op}.

- A category is a graph for which each path of edges is assigned an edge
- That is $\mathcal{C}$ is an algebra for the *paths* monad on $\mathbf{Grph} \cong \mathbf{Set}^{\bullet \rightrightarrows \bullet}$.
- We have $\mathrm{comp}\colon \mathit{paths}(\mathcal{C}) \to \mathcal{C}$. Just like for Lawvere theories:

$$\begin{tikzcd}
1 \arrow[r, "\mathcal{C}"] \arrow[drr, bend right, "\mathcal{C}"'] & \mathbf{Grph} \arrow[rrr, "\mathit{paths}"] \arrow[dr, "\mathit{paths}"] \arrow[d, phantom, "\Downarrow \mathrm{comp}"] & & & \mathbf{Grph} \\
& {} & \mathbf{Grph} \arrow[r, "\mathit{paths}"'] & \mathbf{Grph} \arrow[ur, "{\begin{bmatrix} \mathit{paths}\circ\mathit{paths} \\ \mathit{paths} \end{bmatrix}}"'] \arrow[u, phantom, "\Downarrow"] &
\end{tikzcd}$$

- This makes $\mathit{paths}(\mathcal{C})$ a coalgebra, i.e. a copresheaf, on $\begin{bmatrix} \mathit{paths}\circ\mathit{paths} \\ \mathit{paths} \end{bmatrix}$.
- And indeed, $\begin{bmatrix} \mathit{paths}\circ\mathit{paths} \\ \mathit{paths} \end{bmatrix} \cong \Delta^{\mathrm{op}}$. Its copresheaves are simplicial sets.
- Also $\begin{bmatrix} \mathit{paths} \\ \mathit{paths} \end{bmatrix} \cong \Delta^{\mathrm{op}}_{\mathrm{inert}}$, and the unit induces $\begin{bmatrix} \mathit{paths} \\ \mathit{paths} \end{bmatrix} \xrightarrow{\mathrm{io}} \begin{bmatrix} \mathit{paths}\circ\mathit{paths} \\ \mathit{paths} \end{bmatrix}$

Same idea works for higher cat'l structures like nerves of double cat'ies.

- Pick *cell-shape* category $\mathcal{G}$, *compositions* pra-monad $t : \mathbf{Set}^{\mathcal{G}} \to \mathbf{Set}^{\mathcal{G}}$
- Nerves of that sort of thing are $\begin{bmatrix} t\circ t \\ t \end{bmatrix}$-coalgebras, i.e. copresheaves.

Outline

Summary

Thinking about categories as comonads opens up a new source of cat'ies.

- As density comonads: certain Kan extensions give rise to categories.
- If p, k pra, $t \circ k \to k \circ t$ monad-comonad distr, then $\begin{bmatrix} p \circ t \\ p \circ k \end{bmatrix}$ comonad.
- You can read off the obs and mor's from the defining polynomials.

Summary

Thinking about categories as comonads opens up a new source of cat'ies.

- As density comonads: certain Kan extensions give rise to categories.
- If p, k pra, $t \circ k \to k \circ t$ monad-comonad distr, then $\begin{bmatrix} p \circ t \\ p \circ k \end{bmatrix}$ comonad.
- You can read off the obs and mor's from the defining polynomials.

Lawvere theories, nerve theories, and finite product compl'ns arise this way.

- If you want to see, e.g. monoidal double cat'ies, as presheaves on sth.
- Write out your cell shape cat'y $\mathcal{G}$ and compositions as $t : \mathbb{P}(\mathcal{G}, \mathcal{G})$.
- Then the comonad $\Theta_t^{\mathsf{op}} := \begin{bmatrix} t \circ t \\ t \end{bmatrix}$ is a category.
- Every algebra (monoidal double cat'y) is a Θ_t-presheaf.
- The Lawvere theory $\begin{bmatrix} list \circ t \\ list \end{bmatrix}$ is always a category, since *list* is pra.

Summary

Thinking about categories as comonads opens up a new source of cat'ies.

- As density comonads: certain Kan extensions give rise to categories.
- If p, k pra, $t \circ k \to k \circ t$ monad-comonad distr, then $\begin{bmatrix} p \circ t \\ p \circ k \end{bmatrix}$ comonad.
- You can read off the obs and mor's from the defining polynomials.

Lawvere theories, nerve theories, and finite product compl'ns arise this way.

- If you want to see, e.g. monoidal double cat'ies, as presheaves on sth.
- Write out your cell shape cat'y $\mathcal{G}$ and compositions as $t : \mathbb{P}(\mathcal{G}, \mathcal{G})$.
- Then the comonad $\Theta_t^{\mathsf{op}} := \begin{bmatrix} t \circ t \\ t \end{bmatrix}$ is a category.
- Every algebra (monoidal double cat'y) is a Θ_t-presheaf.
- The Lawvere theory $\begin{bmatrix} \mathit{list} \circ t \\ \mathit{list} \end{bmatrix}$ is always a category, since *list* is pra.

Thank you for your time; questions and comments welcome.

References

- D.I. Spivak, *Categories by Kan extension*. arXiv:2503.21974.
- B.T. Shapiro, D.I. Spivak, *A Polynomial Construction of Nerves for Higher Categories*. arXiv:2405.13157.
- D. Ahman, T. Uustalu, *Directed Containers as Categories*. EPTCS 207 (2016).
- M. Weber, *Familial 2-functors and parametric right adjoints*. TAC 18(22) (2007).
- N. Gambino, J. Kock, *Polynomial functors and polynomial monads*. Math. Proc. Camb. Phil. Soc. 154 (2013).
- F.W. Lawvere, *Functorial semantics of algebraic theories*. PNAS 50 (1963).

Extra slide: proof of theorem 1

Theorem

Suppose $p\colon \mathbf{Set}^{\mathcal{C}} \to \mathbf{Set}^{\mathcal{D}}$ *is a prafunctor.*
Let k *be a pra comonad on* $\mathbf{Set}^{\mathcal{C}}$, *let* t *be any monad on* $\mathbf{Set}^{\mathcal{C}}$, *and let* $t \circ k \xrightarrow{\lambda} k \circ t$ *be a distributive law. Then* $\begin{bmatrix} p \circ t \\ p \circ k \end{bmatrix}$ *is a pra comonad.*

Extra slide: proof of theorem 1

Theorem

Suppose $p\colon \mathbf{Set}^{\mathcal{C}} \to \mathbf{Set}^{\mathcal{D}}$ is a prafunctor.
Let k be a pra comonad on $\mathbf{Set}^{\mathcal{C}}$, let t be any monad on $\mathbf{Set}^{\mathcal{C}}$, and let $t \circ k \xrightarrow{\lambda} k \circ t$ be a distributive law. Then $\begin{bmatrix} p \circ t \\ p \circ k \end{bmatrix}$ is a pra comonad.

- $p \circ k$ is pra and $p \circ t$ is arbitrary, so $\begin{bmatrix} p \circ t \\ p \circ k \end{bmatrix}$ exists and is pra.
- Hence as soon as it's a comonad, it's a cat'y.
- The comonad structure is formal.

Extra slide: proof of theorem 1

Theorem

Suppose $p\colon \mathbf{Set}^{\mathscr{C}} \to \mathbf{Set}^{\mathscr{D}}$ is a prafunctor.
Let k be a pra comonad on $\mathbf{Set}^{\mathscr{C}}$, let t be any monad on $\mathbf{Set}^{\mathscr{C}}$, and let $t \circ k \xrightarrow{\lambda} k \circ t$ be a distributive law. Then $\begin{bmatrix} p\circ t \\ p\circ k \end{bmatrix}$ is a pra comonad.

- $p \circ k$ is pra and $p \circ t$ is arbitrary, so $\begin{bmatrix} p\circ t \\ p\circ k \end{bmatrix}$ exists and is pra.
- Hence as soon as it's a comonad, it's a cat'y.
- The comonad structure is formal.

Strategy: absorb k and t into the functor, i.e. find p' with $\begin{bmatrix} p\circ t \\ p\circ k \end{bmatrix} \cong \begin{bmatrix} p' \\ p' \end{bmatrix}$.

- Now p' won't be pra any more, so hard to work with. But proves thm.
- The only thing we need is duals: if $\ell \dashv r$ then $\begin{bmatrix} q \\ p\circ r \end{bmatrix} \cong \begin{bmatrix} q\circ \ell \\ p \end{bmatrix}$.

Extra slide: proof of theorem 2

Let $\mathbf{Set}^{\mathcal{C}} \xrightarrow{\ell} \mathscr{A} \xrightarrow{r} \mathbf{Set}^{\mathcal{C}}$ be the EM adjunction of t: $\mathscr{A} = t\text{-}\mathbf{Alg}$ and $t{=}r{\circ}\ell$.

- Now rewrite using duals: $\begin{bmatrix} p{\circ}t \\ p{\circ}k \end{bmatrix} = \begin{bmatrix} p{\circ}r{\circ}\ell \\ p{\circ}k \end{bmatrix} \cong \begin{bmatrix} p{\circ}r \\ p{\circ}k{\circ}r \end{bmatrix}$.
- The distr'tive law λ is equivalent to a lift of k to a comonad $\overline{k}$ on $\mathscr{A}$

$$\overline{k}\big(\, t \circ X \xrightarrow{\alpha} X \,\big) \qquad := \qquad t \circ k \circ X \xrightarrow{\lambda \circ X} k \circ t \circ X \xrightarrow{k \circ \alpha} k \circ X.$$

- So $k \circ r = r \circ \overline{k}$, and we get $\begin{bmatrix} p{\circ}t \\ p{\circ}k \end{bmatrix} \cong \begin{bmatrix} p{\circ}r \\ p{\circ}r{\circ}\overline{k} \end{bmatrix}$.

Extra slide: proof of theorem 2

Let $\mathbf{Set}^{\mathcal{C}} \xrightarrow{\ell} \mathcal{A} \xrightarrow{r} \mathbf{Set}^{\mathcal{C}}$ be the EM adjunction of t: $\mathcal{A} = t\text{-}\mathbf{Alg}$ and $t{=}r{\circ}\ell$.

- Now rewrite using duals: $\begin{bmatrix} p{\circ}t \\ p{\circ}k \end{bmatrix} = \begin{bmatrix} p{\circ}r{\circ}\ell \\ p{\circ}k \end{bmatrix} \cong \begin{bmatrix} p{\circ}r \\ p{\circ}k{\circ}r \end{bmatrix}$.
- The distr'tive law λ is equivalent to a lift of k to a comonad $\overline{k}$ on $\mathcal{A}$

$$\overline{k}\big(\, t \circ X \xrightarrow{\alpha} X \,\big) \qquad := \qquad t \circ k \circ X \xrightarrow{\lambda \circ X} k \circ t \circ X \xrightarrow{k \circ \alpha} k \circ X.$$

- So $k \circ r = r \circ \overline{k}$, and we get $\begin{bmatrix} p{\circ}t \\ p{\circ}k \end{bmatrix} \cong \begin{bmatrix} p{\circ}r \\ p{\circ}r{\circ}\overline{k} \end{bmatrix}$.

Now do the same thing with $\overline{k}$: take its EM adjunction.

- Say $\mathcal{B} \xrightarrow{\ell'} \mathcal{A} \xrightarrow{r'} \mathcal{B}$, where $\mathcal{B} = \overline{k}\text{-}\mathbf{Coalg}$ and $\overline{k} = \ell' \circ r'$.
- Rewrite using duals again: $\begin{bmatrix} p{\circ}r \\ p{\circ}r{\circ}\overline{k} \end{bmatrix} = \begin{bmatrix} p{\circ}r \\ p{\circ}r{\circ}\ell'{\circ}r' \end{bmatrix} \cong \begin{bmatrix} p{\circ}r{\circ}\ell' \\ p{\circ}r{\circ}\ell' \end{bmatrix}$.
- That's the density comonad $\begin{bmatrix} p' \\ p' \end{bmatrix}$ for $p' := p \circ r \circ \ell'$.

In fact it's a pra-comonad $\begin{bmatrix} p' \\ p' \end{bmatrix} \cong \begin{bmatrix} p{\circ}t \\ p{\circ}k \end{bmatrix}$, hence a category as desired.